Now available in paperback, this celebrated book has been prepared with readers' needs in mind, giving a systematic treatment of the subject whilst retaining its vitality. The authors' aim is not to present the subject of Brownian motion as a dry part of mathematical analysis, but to convey its real meaning and fascination. The opening, heuristic chapter does just this, and it is followed by a comprehensive and self-contained account of the foundations of the theory of stochastic processes. Chapter III is a lively and readable treatment of the theory of Markov processes. Together with Volume 2: Itô Calculus, this book helps equip graduate students for research into a subject of great intrinsic interest and wide application in physics, biology, engineering, finance and computer science.

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### Some Frequently Used Notation

# **CHAPTER I. BROWNIAN MOTION**

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