Contents

Pr	eface	to the Second Edition	vii			
Pr	eface	to the First Edition	ix			
Ac	know	ledgments to the Second Edition	xi			
Ac	know	ledgments to the First Edition	xiii			
Int	roduc	etion to Rational Points on Plane Curves	1			
	1	Rational Lines in the Projective Plane	2			
	2	Rational Points on Conics	4			
	3	Pythagoras, Diophantus, and Fermat	7			
	4	Rational Cubics and Mordell's Theorem	10			
	5	The Group Law on Cubic Curves and Elliptic Curves	13			
	6	Rational Points on Rational Curves. Faltings and the Mordell				
		Conjecture	17			
	7 8	Real and Complex Points on Elliptic Curves	19			
		in Projective Three Space	20			
1	Elementary Properties of the Chord-Tangent Group Law					
	on a	Cubic Curve	23			
	1	Chord-Tangent Computational Methods on a				
		Normal Cubic Curve	23			
	2	Illustrations of the Elliptic Curve Group Law	28			
	3	The Curves with Equations $y^2 = x^3 + ax$ and $y^2 = x^3 + a$	34			
	4	Multiplication by 2 on an Elliptic Curve	38			
	5	Remarks on the Group Law on Singular Cubics	41			
2	Plan	ne Algebraic Curves	45			
	1	Projective Spaces	45			
	2	Irreducible Plane Algebraic Curves and Hypersurfaces	47			

	XVI	Contents				Contents	xvii
		3 Elements of Intersection Theory for Plane Curves	50		7	The Canonical Height and Norm on an Elliptic Curve	137
		4 Multiple or Singular Points	52	000	8	The Canonical Height on Projective Spaces over Global Fields	
				350	0	The Canonical Height on Projective Spaces over Steam 2 1974	310
		Appendix to Chapter 2: Factorial Rings and Elimination Theory	57	7	G	alois Cohomology and Isomorphism Classification	
		Divisibility Properties of Factorial Rings	57	100	of	f Elliptic Curves over Arbitrary Fields	143
		2 Factorial Properties of Polynomial Rings	59	31.0	1	Galois Theory: Theorems of Dedekind and Artin	143
		3 Remarks on Valuations and Algebraic Curves	60	380	2	Group Actions on Sets and Groups	146
		4 Resultant of Two Polynomials	61	000	3	Principal Homogeneous G-Sets and the First Cohomology Set	
		*		OPS		$H^1(G,A)$	148
3	3	Elliptic Curves and Their Isomorphisms	65	253	4	Long Exact Sequence in G-Cohomology	151
		1 The Group Law on a Nonsingular Cubic	65	520	5	Some Calculations with Galois Cohomology	153
	- 1	2 Normal Forms for Cubic Curves	67	225	6	Galois Cohomology Classification of Curves with Given j-Invariant	155
-		The Discriminant and the Invariant <i>j</i>	70	02.0		The Consecues to keeinGrand let when head 24 a none shall	
		4 / Isomorphism Classification in Characteristics $\neq 2, 3$	73	8	D	escent and Galois Cohomology	157
		5 Isomorphism Classification in Characteristic 3	75	285	1	Homogeneous Spaces over Elliptic Curves	157
		6 Isomorphism Classification in Characteristic 2	76	286	2	Primitive Descent Formalism	160
		7 Singular Cubic Curves	80	250	3	Basic Descent Formalism	163
		8 Parameterization of Curves in Characteristic Unequal to 2 or 3	82	270			
				9	E	Illiptic and Hypergeometric Functions	167
4		Families of Elliptic Curves and Geometric Properties			1	Quotients of the Complex Plane by Discrete Subgroups	167
		of Torsion Points	85	275	2		
		The Legendre Family	85		3	The Weierstrass &-Function	
		2 Families of Curves with Points of Order 3: The Hessian Family	88	275	4	The Differential Equation for $\wp(z)$	174
		3 The Jacobi Family	91	577	5		
		4 Tate's Normal Form for a Cubic with a Torsion Point	92	085	6	Periods Associated with Elliptic Curves: Elliptic Integrals	183
		5 An Explicit 2-Isogeny	95	182			
	1	6 Examples of Noncyclic Subgroups of Torsion Points	101	10	T	heta Functions	
5	1	Doduction made and Tout D		195	1	Jacobi q-Parametrization: Application to Real Curves	189
3		Reduction mod p and Torsion Points	103		2		
		and Curves	103	162	3		
		The state of the sample carve	106	293	4	A	
			109		5		
		and the p-Adic Filliation	111	968	6	Introduction to Tate's Theory of <i>p</i> -Adic Theta Functions	203
		2 Trugen Luiz Hichelm	115	11		redmild to anniugeneous at A. B. a. and pages for the dismost. 4	200
	,	Total in the state of the state		11		Modular Functions	
	-	and Divisibility Properties of Coordinates	18		1		
		Bad Reduction and Potentially Good Reduction	.20	168	3		
		rate's Theorem on Good Reduction over the Rational Numbers 1	.22		4		
6	I	Proof of Mordell's Finite Generation Theorem	25	3030	5		
	1	A Condition for Finite Generation of an Abelian Group	25				
	2	Fermat Descent and $x^4 + y^4 = 1$	27		7		
	. 3	Finiteness of $(E(\mathbb{Q}): 2E(\mathbb{Q}))$ for $E = E[a, b]$	28		8		
	4	Finiteness of the Index $(E(k): 2E(k))$.	20		9		
	5	Quasilinear and Quasiquadratic Maps	32		9	Modular Foryholinais and the Modular Equation	THE
	6	The General Notion of Height on Projective Space	35				5
		and the state of the second and the	55			NAS AND THE PROPERTY OF THE PR	FA
						~	INFO
							- kn

Contents

12	End		233
	1	Isogenies and Division Points for Complex Tori	133
	2	Symplectic Pairings on Lattices and Division Points	
	3	Isogenies in the General Case	
	4	Endomorphisms and Complex Multiplication	41
	5	The Tate Module of an Elliptic Curve	145
	6	Endomorphisms and the Tate Module	46
	7	Expansions Near the Origin and the Formal Group	48
13	Ellip	otic Curves over Finite Fields	253
	1	The Riemann Hypothesis for Elliptic Curves over a Finite Field 2	253
ed :	2	Generalities on Zeta Functions of Curves over a Finite Field 2	256
ъ 1	3	Definition of Supersingular Elliptic Curves	159
	4 1	Number of Supersingular Elliptic Curves	63
	5	Points of Order p and Supersingular Curves	65
	6	The Endomorphism Algebra and Supersingular Curves	66
	7	Summary of Criteria for a Curve To Be Supersingular	68
	8	Tate's Description of Homomorphisms	
	9	Division Polynomial	72
14 Elliptic Curves over Local Fields			
	1	The Canonical p-Adic Filtration on the Points of an Elliptic Curve	
		over a Local Field	75
	2	The Néron Minimal Model	77
	3	Galois Criterion of Good Reduction of Néron-Ogg-Šafarevič 2	80
	4	Elliptic Curves over the Real Numbers	
15	Ellip	otic Curves over Global Fields and ℓ -Adic Representations 2	91
	1	Minimal Discriminant Normal Cubic Forms	
		over a Dedekind Ring	91
	2	Generalities on ℓ-Adic Representations	.93
	3	Galois Representations and the Néron-Ogg-Šafarevič Criterion in	
		the Global Case	96
	4	Ramification Properties of \ell-Adic Representations of Number	
		Fields: Čebotarev's Density Theorem	98
	5	Rationality Properties of Frobenius Elements in ℓ-Adic	
		Representations: Variation of ℓ	01
	6	Weight Properties of Frobenius Elements in ℓ-Adic	
		Representations: Faltings' Finiteness Theorem	
	7	Tate's Conjecture, Šafarevič's Theorem, and Faltings' Proof 3	05
	8	Image of ℓ-Adic Representations of Elliptic Curves: Serre's Open	
		Image Theorem	07

		unction of an Elliptic Curve and Its Analytic Continuation 309
6	L-F	Remarks on Analytic Methods in Arithmetic
	1.	Remarks on Analytic Methods in Arithmete. Zeta Functions of Curves over \mathbb{Q} . 310
	2	Zeta Functions of Curves over Q
	3	Classical Abelian <i>L</i> -Functions and Their Functional Equations
	4	Grössencharacters and Hecke <i>L</i> -Functions
	5	Grössencharacters and Hecke L-Functions.
	6	Deuring's Theorem on the <i>L</i> -Function of an Elliptic Curve with 321
		Complex Multiplication
	7	Eichler–Shimura Theory 322
	8	The Modular Curve Conjecture
17	Re	marks on the Birch and Swinnerton–Dyer Conjecture
0	1	The Conjecture Relating Rank and Order of Zero 323
	2	Pank Conjecture for Curves with Complex Multiplication 1, by
	_	Coates and Wiles
	3	Perk Conjecture for Curves with Complex Multiplication II, by
		Country and Pobrlich
	4	Rank Conjecture for Modular Curves by Gross and Zagier 328
	5	Goldfold's Work on the Class Number Problem and its Relation to
		the Rirch and Swinnerton-Dver Conjecture
	6	The Conjecture of Rirch and Swinnerton-Dyer on the Leading 1erm 329
	7	Heagner Points and the Derivative of the L-function at $s=1$, after
		Cross and Zagier
	8	Remarks On Postscript: October 1986
18	R	emarks on the Modular Elliptic Curves Conjecture and
	F	armot's Lost Theorem
	1	Samietable Curves and Tate Modules
	2	The Frey Curve and the Reduction of Fermat Equation to Modular
		Elliptic Curves over (1)
	3	Modular Elliptic Curves and the Hecke Algebra
	4	Hecke Algebras and Tate Modules of Modular Elliptic Curves 338
	5	Special Properties of mod 3 Representations
	6	Deformation Theory and ℓ -Adic Representations
	7	Properties of the Universal Deformation Ring
	8	Remarks on the Proof of the Opposite Inequality
	9	Survey of the Nonsemistable Case of the Modular Curve Conjecture 342
19	9 F	ligher Dimensional Analogs of Elliptic Curves:
1	Ĉ	Calabi Van Varieties
	1	Smooth Manifolds: Real Differential Geometry
	2	Complex Analytic Manifolds: Complex Differential Geometry 349
	3	Kähler Manifolds
	4	Connections Curvature, and Holonomy
	-	and the contraction of the contr

Conter	

	6 Char	racterizations of Calabi-Yau Manifolds: First Examples
	7 Exar	nples of Calabi–Yau Varieties from Toric Geometry 369
	8 Line	Bundles and Divisors: Picard and Néron–Severi Groups 371
	9 Num	perical Invariants of Surfaces
	10 Enric	ques Classification for Surfaces
	11 Intro	duction to K3 Surfaces
20	Families o	of Elliptic Curves
	1 Alge	braic and Analytic Geometry
	2 Morr	phisms Into Projective Spaces Determined by Line Bundles,
	Divis	sors, and Linear Systems
	3 Fibra	ations Especially Surfaces Over Curves
	4 Gene	eralities on Elliptic Fibrations of Surfaces Over Curves 392
	5 Ellip	tic K3 Surfaces
	6 Fibra	ations of 3 Dimensional Calabi–Yau Varieties
		e Examples of Three Dimensional Calabi–Yau Hypersurfaces
	in W	11.5 1 1 5 6 4 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1
	III VV	eight Projective Four Space and Their Fibrings
Apr	endix I: Ca	alabi-Yau Manifolds and String Theory 403
	Stefan The	icen
		g Theory?
		erties
		ories in Ten Dimensions
		cation
		409
	Summary	
	11 YY Y	
App		Elliptic Curves in Algorithmic Number Theory and
		pphy
	Otto Forste	
	1 Appli	ications in Algorithmic Number Theory
	1.1	Factorization
	1.2	Deterministic Primality Tests
	2 Ellipt	ic Curves in Cryptography
	2.1	The Discrete Logarithm
	2.2	Diffie–Hellman Key Exchange
	2.3	Digital Signatures
	2.4	Algorithms for the Discrete Logarithm
	2.5	Counting the Number of Points
	2.6	Schoof's Algorithm
	2.7	Elkies Primes 423
		2424 424
	- LUICION CHICCS	

	Contents x	X1
ppendi	x III: Elliptic Curves and Topological Modular Forms	
1	Categories in a Category	27
2	Groupoids in a Category	29
3	Cocategories over Commutative Algebras: Hopf Algebroids 43	31
4	The Category $WT(R)$ and the Weierstrass Hopf Algebroid 43	34
5	Morphisms of Hopf Algebroids: Modular Forms	38
6	The Role of the Formal Group in the Relation Between Elliptic	
0	Curves and General Cohomology Theory 44	41
7	The Cohomology Theory or Spectrum tmf	43
	erences	44
Rei	erences	
ppend	ix IV: Guide to the Exercises4	45
	h Lawrence	
eferen	ces	65
ist of N	Notation	79
	Upper state of the	81