Mathematicians have only recently begun to understand the local structure of solutions of degenerate and singular parabolic partial differential equations. The problem originated in the mid 1960s with the work of DeGiorgi, Moser, Ladyzenskajia and Ural'tzeva. This book is an account of the developments in this field over the past five years. It evolved out of the 1990 Lipschitz Lectures given by Professor DiBenedetto at the *Institut für Angewandte Mathematik* of the University of Bonn.

ISBN 978-0-387-94020-5

9 780387 940205

ISBN 978-0-387-94020-5

Preface

§1. Elliptic equations: Harnack estimates and Hölder continuity v
§2. Parabolic equations: Harnack estimates and Hölder continuity vi
§3. Parabolic equations and systems vii
§4. Main results viii

I. Notation and function spaces

§1. Some notation 1
§2. Basic facts about $W^{1,p}(\Omega)$ and $W_o^{1,p}(\Omega)$ 3
§3. Parabolic spaces and embeddings 7
§4. Auxiliary lemmas 12
§5. Bibliographical notes 15

II. Weak solutions and local energy estimates

§1. Quasilinear degenerate or singular equations 16
§2. Boundary value problems 20
§3. Local integral inequalities 22
§4. Energy estimates near the boundary 31
§5. Restricted structures: the levels k and the constant γ 38
§6. Bibliographical notes 40

III. Hölder continuity of solutions of degenerate parabolic equations

§1. The regularity theorem 41
§2. Preliminaries 43
§3. The main proposition 44
§4. The first alternative 49
§5. The first alternative continued 52
§6. The first alternative concluded 55
§7. The second alternative 58
§8. The second alternative continued 62
§9. The second alternative concluded 64
§10. Proof of Proposition 3.1 68
§11. Regularity up to $t = 0$ 69
§12. Regularity up to S_T. Dirichlet data 72
§13. Regularity at S_T. Variational data 74
§14. Remarks on stability 74
§15. Bibliographical notes 75

IV. Hölder continuity of solutions of singular parabolic equations

§1. Singular equations and the regularity theorems 77
§2. The main proposition 79
§3. Preliminaries 81
§4. Rescaled iterations 84
§5. The first alternative 88
§6. Proof of Lemma 5.1. Integral inequalities 92
§7. An auxiliary proposition 95
§8. Proof of Proposition 7.1 when (7.6) holds 97
§9. Removing the assumption (6.1) 101
§10. The second alternative 102
§11. The second alternative concluded 106
§12. Proof of the main proposition 109
§13. Boundary regularity 110
§14. Miscellaneous remarks 114
§15. Bibliographical notes 116

V. Boundedness of weak solutions

§1. Introduction 117
§2. Quasilinear parabolic equations 118
§3. Sup-bounds 120
§4. Homogeneous structures. The degenerate case $p > 2$ 122
§5. Homogeneous structures. The singular case $1 < p < 2$ 125
§6. Energy estimates 128

§7. Local iterative inequalities 131
§8. Local iterative inequalities $\left(p > \max\left\{1; \frac{2N}{N+2}\right\}\right)$ 134
§9. Global iterative inequalities 135
§10. Homogeneous structures and $1<p\leq\max\left\{1; \frac{2N}{N+2}\right\}$ 137
§11. Proof of Theorems 3.1 and 3.2 138
§12. Proof of Theorem 4.1 140
§13. Proof of Theorem 4.2 142
§14. Proof of Theorem 4.3 143
§15. Proof of Theorem 4.5 144
§16. Proof of Theorems 5.1 and 5.2 147
§17. Natural growth conditions 149
§18. Bibliographical notes 155

VI. Harnack estimates: the case $p>2$

§1. Introduction 156
§2. The intrinsic Harnack inequality 157
§3. Local comparison functions 159
§4. Proof of Theorem 2.1 163
§5. Proof of Theorem 2.2 167
§6. Global versus local estimates 169
§7. Global Harnack estimates 171
§8. Compactly supported initial data 172
§9. Proof of Proposition 8.1 174
§10. Proof of Proposition 8.1 continued 177
§11. Proof of Proposition 8.1 concluded 179
§12. The Cauchy problem with compactly supported initial data 180
§13. Bibliographical notes 183

VII. Harnack estimates and extinction profile for singular equations

§1. The Harnack inequality 184
§2. Extinction in finite time (bounded domains) 188
§3. Extinction in finite time (in $\mathbf{R}^N$) 191
§4. An integral Harnack inequality for all $1<p<2$ 193
§5. Sup-estimates for $\frac{2N}{N+1}<p<2$ 198
§6. Local subsolutions 199
§7. Time expansion of positivity 203
§8. Space-time configurations 204
§9. Proof of the Harnack inequality 206
§10. Proof of Theorem 1.2 211
§11. Bibliographical notes 214

VIII. Degenerate and singular parabolic systems

§1. Introduction 215
§2. Boundedness of weak solutions 218
§3. Weak differentiability of $|D\mathbf{u}|^{\frac{p-2}{2}}D\mathbf{u}$ and energy estimates for $|D\mathbf{u}|$ 223
§4. Boundedness of $|D\mathbf{u}|$. Qualitative estimates 231
§5. Quantitative sup-bounds of $|D\mathbf{u}|$ 238
§6. General structures 243
§7. Bibliographical notes 244

IX. Parabolic p-systems: Hölder continuity of $D\mathbf{u}$

§1. The main theorem 245
§2. Estimating the oscillation of $D\mathbf{u}$ 248
§3. Hölder continuity of $D\mathbf{u}$ (the case $p>2$) 251
§4. Hölder continuity of $D\mathbf{u}$ (the case $1<p<2$) 256
§5. Some algebraic Lemmas 258
§6. Linear parabolic systems with constant coefficients 263
§7. The perturbation lemma 268
§8. Proof of Proposition 1.1-(i) 275
§9. Proof of Proposition 1.1-(ii) 278
§10. Proof of Proposition 1.1-(iii) 282
§11. Proof of Proposition 1.1 concluded 284
§12. Proof of Proposition 1.2-(i) 286
§13. Proof of Proposition 1.2 concluded 288
§14. General structures 291
§15. Bibliographical notes 291

X. Parabolic p-systems: boundary regularity

§1. Introduction 292
§2. Flattening the boundary 294
§3. An iteration lemma 297
§4. Comparing $\mathbf{w}$ and $\mathbf{v}$ (the case $p>2$) 299
§5. Estimating the local average of $|D\mathbf{w}|$ (the case $p>2$) 304
§6. Estimating the local averages of $\mathbf{w}$ (the case $p>2$) 305
§7. Comparing $\mathbf{w}$ and $\mathbf{v}$ (the case $\max\left\{1;\frac{2N}{N+2}\right\}<p<2$) 309
§8. Estimating the local average of $|D\mathbf{w}|$ 313
§9. Bibliographical notes 315

XI. Non-negative solutions in Σ_T. The case $p>2$

§1. Introduction 316
§2. Behaviour of non-negative solutions as $|x|\to\infty$ and as $t\searrow 0$ 317
§3. Proof of (2.4) 319
§4. Initial traces 322

§5. Estimating $|Du|^{p-1}$ in Σ_T 323
§6. Uniqueness for data in $L^1_{loc}(\mathbf{R}^N)$ 326
§7. Solving the Cauchy problem 330
§8. Bibliographical notes 333

XII. Non-negative solutions in Σ_T. The case $1<p<2$

§1. Introduction 334
§2. Weak solutions 337
§3. Estimating $|Du|$ 340
§4. The weak Harnack inequality and initial traces 344
§5. The uniqueness theorem 346
§6. An auxiliary proposition 350
§7. Proof of the uniqueness theorem 362
§8. Solving the Cauchy problem 362
§9. Compactness in the space variables 363
§10. Compactness in the t variable 366
§11. More on the time–compactness 370
§12. The limiting process 371
§13. Bounded solutions. A counterexample 376
§14. Bibliographical notes 379

Bibliography 381