

Harnack's Inequality for Degenerate and Singular Parabolic Equations

While degenerate and singular parabolic equations have been researched extensively for the last 25 years, the Harnack inequality for nonnegative solutions to these equations has received relatively little attention. Recent progress has been made on the Harnack inequality to the point that the theory is now reasonably complete—except for the singular subcritical range—both for the p -Laplacian and the porous medium equations.

This monograph provides a comprehensive overview of the subject that highlights open problems. The authors treat the Harnack inequality for nonnegative solutions to p -Laplace and porous medium type equations, both in the degenerate and in the singular range. The work is mathematical in nature; its aim is to introduce a novel set of tools and techniques that deepen our understanding of the notions of degeneracy and singularity in partial differential equations.

Although related in spirit to a monograph by the first author in this subject, this book is a self-contained treatment with a different perspective. Here the focus is entirely on the Harnack estimates and on their applications; the authors use the Harnack inequality to reprove a number of known regularity results. This book is aimed at researchers and advanced graduate students who work in this fascinating field.

Mathematics

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Preface	xiii
1 Introduction	1
1 The Classical Harnack Inequality	1
2 Quasilinear Coercive Elliptic and Parabolic Equations	4
3 Degenerate and Singular Parabolic Equations	5
3.1 Quasilinear Equations of p -Laplacian Type	5
3.2 Quasilinear Equations of Porous Medium Type	6
3.3 Aim of the Monograph	6
4 Parabolic Harnack Estimates. The Role of the Structure	7
4.1 Degenerate Equations of the p -Laplacian Type for $p > 2$	8
4.2 Singular Equations of the p -Laplacian Type for $\frac{2N}{N+1} < p < 2$	9
4.3 Outstanding Issues	10
2 Preliminaries	13
1 Poincaré and Sobolev Inequalities	13
2 Cuts and Truncations of Functions in $W^{1,p}(E)$ and Their Embeddings	15
3 A Measure-Theoretical Lemma ([48])	16
4 Parabolic Spaces and Embeddings	18
5 Some Technical Facts	20
5.1 A Lemma on Fast Geometric Convergence	20
5.2 An Interpolation Lemma	21
5.3 Steklov Averages	21
6 Remarks and Bibliographical Notes	21
3 Degenerate and Singular Parabolic Equations	23
1 Quasilinear Equations of p -Laplacian Type	23
1.1 An Alternate Formulation in Terms of Steklov Averages	24
1.2 On the Notion of Parabolicity	24

1.3	Dependence on the Parameters $\{p, N, C_o, C_1\}$ and Stability	26
2	Energy Estimates for $(u - k)_\pm$ on Cylinders $(y, s) + Q_\rho^\pm(\theta) \subset E_T$	26
3	A DeGiorgi-Type Lemma	29
4	A Variant of DeGiorgi-Type Lemma Involving "Initial Data" ..	32
5	Quasilinear Equations of the Porous Medium Type	33
5.1	An Alternate Formulation in Terms of Steklov Averages	34
5.2	On the Notion of Parabolicity	35
5.3	Dependence on the Parameters $\{m, N, C_o, C_1\}$ and Stability	36
6	Energy Estimates for $(u - k)_\pm$ on Cylinders $(y, s) + Q_\rho^\pm(\theta) \subset E_T$ for Degenerate Equations $(m > 1)$	36
7	A DeGiorgi-Type Lemma for Nonnegative Sub(Super)-Solutions to Degenerate Equations $(m > 1)$	39
7.1	Proof of (7.1)–(7.3)	40
7.2	Proof of (7.4)–(7.5)	42
8	A Variant of DeGiorgi-Type Lemma, for Nonnegative Supersolutions to Degenerate Equations $(m > 1)$, Involving "Initial Data"	45
9	Energy Estimates for $(u - k)_-$ on Cylinders $(y, s) + Q_\rho^\pm(\theta) \subset E_T$ for Singular Equations $(0 < m < 1)$	46
10	A DeGiorgi-Type Lemma for Nonnegative Supersolutions to Singular Equations $(0 < m < 1)$	50
11	A Variant of DeGiorgi-Type Lemma, for Nonnegative Supersolutions to Singular Equations $(0 < m < 1)$, Involving "Initial Data"	52
12	Remarks and Bibliographical Notes	54
4	Expansion of Positivity	57
1	Time and Space Propagation of Positivity	57
2	The Expansion of Positivity for Nondegenerate, Homogeneous, Quasilinear Parabolic Equations	59
3	Some Counterexamples for Degenerate and Singular Equations	62
3.1	A First Counterexample for $p > 2$	62
3.2	A Second Counterexample for $p > 2$	63
3.3	A Family of Counterexamples for $1 < p < 2$	64
3.4	The Expansion of Positivity in Some Intrinsic Geometry	66
4	The Expansion of Positivity for Degenerate Quasilinear Parabolic Equations $(p > 2)$	66
4.1	Structure of the Proof	67
4.2	Changing the Time Variable	67
4.3	The Set Where w Is Small Can Be Made Small Within $Q_{4\rho}(\theta)$ for Large θ	69
4.4	Expanding the Positivity of w	71

4.5	Expanding the Positivity of u	72
5	The Expansion of Positivity for Singular Quasilinear Parabolic Equations ($1 < p < 2$)	72
5.1	Transforming the Variables and the Equation	73
5.2	Estimating the Measure of the Set $[v < k_j]$ Within Q_{τ_o} ..	75
5.3	Segmenting Q_{τ_o}	76
5.4	Returning to the Original Coordinates	77
6	Stability of the Expansion of Positivity for $p \rightarrow 2$	78
6.1	Proof of Proposition 6.1	79
7	The Expansion of Positivity for Porous Medium Type Equations	81
7.1	Expansion of Positivity When $m > 1$	82
7.2	Expansion of Positivity When $0 < m < 1$	83
8	Remarks and Bibliographical Notes	88
8.1	Solving (3.6)	89
5	The Harnack Inequality for Degenerate Equations	91
1	The Intrinsic Harnack Inequality	91
1.1	The Mean Value Form of Moser's Harnack Inequality ..	91
1.2	The Intrinsic Mean Value Harnack Inequality for Degenerate Equations	92
1.3	Significance of Theorem 1.1	94
2	Proof of the Intrinsic, Forward Harnack Inequality in Theorem 1.1	95
2.1	Proof of Proposition 2.1	96
2.2	Expanding the Positivity of v	97
2.3	Proof of Proposition 2.1 Concluded	99
2.4	On the Connection Between γ and c	100
3	Proof of the Intrinsic, Backward Harnack Inequality in Theorem 1.1	100
3.1	There Exists $t < t_o$ Satisfying (3.1)	101
4	The Intrinsic Harnack Inequality Implies the Hölder Continuity ..	102
5	Liouville-Type Results	105
5.1	Two-Sided Bounds and Liouville-Type Results	106
5.2	Two-Sided Bound at One Point, as $t \rightarrow \infty$	107
6	Subpotential Lower Bounds	108
7	A Weak Harnack Inequality for Positive Supersolutions	112
8	Proof of Proposition 7.1	113
9	Proof of Theorem 7.1 by Alternatives	115
10	A Reverse Hölder Inequality for Supersolutions	116
11	A Uniform Bound Above on the L^q_{loc} -Integral of Supersolutions ..	120
12	An Integral Bound Below for Supersolutions	121
13	Proof of Theorem 7.1	123
14	A Consequence of Theorem 7.1	125
15	Equations of the Porous Medium Type	127

15.1	The Intrinsic Harnack Inequality	127
16	Some Consequences of the Harnack Inequality	130
16.1	Hölder Continuity	130
16.2	Liouville-Type Results	131
16.3	Subpotential Lower Bounds	132
17	A Weak Harnack Inequality for Positive Supersolutions	133
18	Remarks and Bibliographical Notes	134
18.1	Weak Harnack Estimates	136
6	The Harnack Inequality for Singular Equations	137
1	Supercritical, Singular Equations	137
1.1	The Intrinsic, Mean Value, Harnack Inequality	137
1.2	Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities	139
1.3	On the Range (1.1) of p	140
2	Main Components in the Proof of Theorems 1.1 and 1.2	142
3	The Right-Hand-Side Harnack Estimate of Theorem 1.2	143
4	Locating the Supremum of v in K_1	144
5	Estimating the Sup of v in Some Intrinsic Neighborhood About $(\bar{x}, 0)$	144
6	Expanding the Positivity of v	146
7	Proof of the Right-Hand-Side Harnack Inequality of Theorem 1.1	148
7.1	On the Functional Relation $\gamma = \gamma(\epsilon)$	150
8	Proof of the Left-Hand-Side Harnack Inequality in Theorem 1.2 ..	151
9	Proof of the Left-Hand-Side Harnack Inequality in Theorem 1.1 ..	151
10	Some Consequences of the Harnack Inequality	153
10.1	Local Hölder Continuity	153
10.2	A Liouville-Type Result	153
11	Critical and Subcritical Singular Equations	155
12	Components of the Proof of Theorem 11.1	156
12.1	$L_{\text{loc}}^r - L_{\text{loc}}^\infty$ Harnack-Type Estimates for $r \geq 1$ Such That $\lambda_r > 0$	156
12.2	L_{loc}^r Estimates Backward in Time	157
13	Estimating the Positivity Set of the Solutions	157
13.1	A First Form of the Harnack Inequality	159
14	The First Form of the Harnack Inequality Implies the Hölder Continuity of u	160
15	Proof of Theorem 11.1 Concluded	163
16	Supercritical, Singular Equations of the Porous Medium Type ..	165
16.1	The Intrinsic, Mean Value, Harnack Inequality	165
16.2	Time-Insensitive, Intrinsic, Mean Value, Harnack Inequalities	167
16.3	On the Range (16.1) of m	168
17	On the Proof of Theorems 16.1 and 16.2	170

17.1	An $L^1_{\text{loc}}-L^\infty_{\text{loc}}$ Harnack-Type Estimate	170
17.2	The Right-Hand-Side Harnack Estimate of Theorem 16.2	170
18	Some Consequences of the Harnack Inequality	171
18.1	Analyticity of Nonnegative Solutions	171
18.2	Hölder Continuity	172
18.3	A Liouville-Type Result	175
19	Critical and Subcritical Singular Equations of the Porous Medium Type	175
20	On the Proof of Theorem 19.1	177
20.1	$L^r_{\text{loc}}-L^\infty_{\text{loc}}$ Harnack-Type Estimates for $r \geq 1$ Such That $\lambda_r > 0$	177
20.2	L^r_{loc} Estimates Backward in Time	177
20.3	Main Points of the Proof	178
21	Remarks and Bibliographical Notes	179
21.1	About Theorem 11.1	180
21.2	On the Weak Harnack Inequality	181
21.3	On the Boundedness of Weak Solutions	181
21.4	On the Two Critical Values $p_{**} = \frac{2N}{N+2} < \frac{2N}{N+1} = p_*$	182
21.5	Singular Equations of the Porous Medium Type	183
21.6	On the Two Critical Values $0 \leq m_{**} \leq m_* < 1$	184
7	Homogeneous Monotone Singular Equations	187
1	Monotone Structure Conditions	187
1.1	A Less Constrained Harnack Inequality	189
2	The Intrinsic Harnack Inequality	190
2.1	The Right-Hand-Side Harnack Estimate of Theorem 2.2	191
3	Proof of Proposition 2.1	191
3.1	Largeness of v in Q_1	192
3.2	Expanding the Positivity of v	193
4	Subpotential Lower Bounds	196
5	Monotone Structures for Singular Porous Medium Type Equations	199
5.1	A Less Constrained Harnack Inequality	201
6	The Intrinsic Harnack Inequality	202
6.1	The Right-Hand-Side Harnack Estimate of Theorem 2.2	203
6.2	Subpotential Lower Bounds	203
7	Remarks and Bibliographical Notes	204
8	Appendix A	205
A.1	An L^1_{loc} -Form of the Harnack Inequality for All $p \in (1, 2)$	205
A.1.1	Auxiliary Lemmas	205
A.1.2	Proof of Proposition A.1.1	208
A.1.3	An Estimate for Supersolutions	210
A.2	$L^r_{\text{loc}}-L^\infty_{\text{loc}}$ Estimates	210

A.2.1	Proof of Proposition A.2.1 for p in the Range $\max\{1, \frac{2N}{N+2}\} < p < 2$	211
A.2.2	Proof of Proposition A.2.1 for p in the Range $1 < p \leq \max\{1, \frac{2N}{N+2}\}$	213
A.3	L^r_{loc} Estimates Backward in Time	215
A.3.1	Proof of Proposition A.3.1	215
A.4	Remarks and Bibliographical Notes	219
9	Appendix B	221
B.1	An L^1_{loc} -Form of the Harnack Inequality for All $m \in (0, 1)$	221
B.1.1	An Auxiliary Lemma	221
B.1.2	Proof of Proposition B.1.1	223
B.1.3	An Estimate for Supersolutions	224
B.2	Energy Estimates for Sub(Super)-Solutions When $0 < m < 1$..	225
B.3	A Different Type of Energy Estimates for Sub(Super)-Solutions When $0 < m < 1$	227
B.4	$L^r_{\text{loc}}-L^\infty_{\text{loc}}$ Estimates	230
B.4.1	Proof of Proposition B.4.1 for $\frac{(N-2)_+}{N+2} < m < 1$	231
B.4.2	Proof of Proposition B.4.1 for $0 < m \leq \frac{(N-2)_+}{N+2}$	234
B.5	L^r_{loc} Estimates Backward in Time	235
B.6	A DeGiorgi-Type Lemma for Nonnegative Subsolutions to Singular Porous Medium Type Equations	238
B.7	A Logarithmic Estimate for Nonnegative Subsolutions to Singular Porous Medium Type Equations	240
B.8	Hölder Continuity	243
B.8.1	Proof of Theorem B.8.1. Preliminaries	243
B.9	The Induction Argument. Two Alternatives	244
B.10	A Uniform Time Control on the Measure of the Level Sets	245
B.11	The Second Alternative Continued	248
B.12	The Induction Argument Concluded-(i)	250
B.13	The Induction Argument Concluded-(ii)	251
B.14	Remarks and Bibliographical Notes	252
10	Appendix C	255
C.1	More General Structures	255
C.2	Energy Estimates for $(u-k)_\pm$ on Cylinders $(y, s) + Q^\pm_\rho(\theta) \subset E_T$..	256
C.3	DeGiorgi-Type Lemmas	257
C.3.1	A Variant of DeGiorgi-Type Lemma, Involving "Initial Data"	258
C.3.2	A Technical Lemma	258
C.4	Expansion of Positivity for Nonnegative Solutions to (C.1.1)–(C.1.4)	259
C.5	Equations of Porous Medium Type	261
C.6	Energy Estimates for $(u-k)_\pm$ on Cylinders $(y, s) + Q^\pm_\rho(\theta) \subset E_T$..	261
C.7	DeGiorgi-Type Lemmas	263

C.7.1	A Variant of DeGiorgi-Type Lemma, Involving “Initial Data”	264
C.8	Expansion of Positivity for Solutions to (C.5.1)–(C.5.3)	265
C.9	Remarks and Bibliographical Notes	266
References		269
Index		277