CONTENTS

Preface ix

1 Foundations for Learning Mathematics 1

EXPLORATION 1.1 Patterns, Problem Solving, and Representations 2
EXPLORATION 1.2 Math Knowledge for Teaching 6
EXPLORATION 1.3 Magic Triangle Puzzles 7
EXPLORATION 1.4 Patterns and Proof 9
EXPLORATION 1.5 Real-life Problems 12
EXPLORATION 1.6 Magic Squares 13
EXPLORATION 1.7 Patterns in Multiplication 16

2 Fundamental Concepts 19

2.1 • Exploring Sets 19

EXPLORATION 2.1 Understanding Venn Diagrams 20
EXPLORATION 2.2 Gathering and Interpreting Data 21

2.2 • Exploring Numeration 22

EXPLORATION 2.3 Alphabitia 22
EXPLORATION 2.4 Different Bases 27
EXPLORATION 2.5 A Place Value Game 30
EXPLORATION 2.6 How Big Is Big? 31

3 The Four Fundamental Operations of Arithmetic 33

EXPLORATION 3.1 Computation in Alphabitia 33

3.1 • Exploring Addition 35

EXPLORATION 3.2 Mental Addition 35
EXPLORATION 3.3 Addition: Children's Algorithms and Alternative Algorithms 36
EXPLORATION 3.4 Addition and Number Sense 38

3.2 • Exploring Subtraction 39

EXPLORATION 3.5 Mental Subtraction 39
EXPLORATION 3.6 Subtraction: Children's Algorithms and Alternative Algorithms 40
EXPLORATION 3.7 Subtraction and Number Sense 42

3.3 • Exploring Multiplication 43

EXPLORATION 3.8 Patterns in the Multiplication Table 43
EXPLORATION 3.9 Mental Multiplication 47

EXPLORATION 3.10 Differences Between Multiplication and Addition 48
EXPLORATION 3.11 Cluster or String Problems 49
EXPLORATION 3.12 Understanding the Standard Multiplication Algorithm 50
EXPLORATION 3.13 Alternative Algorithms for Multiplication 51
EXPLORATION 3.14 Multiplication and Number Sense 53

3.4 • Exploring Division 54

EXPLORATION 3.15 Different Models of Division 54
EXPLORATION 3.16 Understanding Division Computation 55
EXPLORATION 3.17 Mental Division 56
EXPLORATION 3.18 Dealing with Remainders 57
EXPLORATION 3.19 The Scaffolding Algorithm 58
EXPLORATION 3.20 Understanding the Standard Algorithm 59
EXPLORATION 3.21 Division and Number Sense 60
EXPLORATION 3.22 Developing Operation Sense 61
EXPLORATION 3.23 Operation Sense in Games 62
EXPLORATION 3.24 Taxman 63
EXPLORATION 3.25 Factors 67
EXPLORATION 3.26 How Many Stars? 73

4 **Extending the Number System 75**

4.1 • Exploring Integers 75

EXPLORATION 4.1 Understanding Integer Addition 76
EXPLORATION 4.2 Understanding Integer Subtraction 77
EXPLORATION 4.3 Understanding Integer Multiplication 78
EXPLORATION 4.4 Understanding Integer Division 79

4.2 • Exploring Fractions and Rational Numbers 80

EXPLORATION 4.5 Making Manipulatives 80
EXPLORATION 4.6 Sharing Brownies 81
EXPLORATION 4.7 Partitioning 82
EXPLORATION 4.8 Equivalent Fractions 87
EXPLORATION 4.9 Developing Fraction Sense 88

4.3 • Exploring Operations with Fractions 89

EXPLORATION 4.10 Finding All Factors of a Number 89
EXPLORATION 4.11 African Sand Drawings 90
EXPLORATION 4.12 Cycles 92
EXPLORATION 4.13 Ordering Fractions 94
EXPLORATION 4.14 Adding Fractions 97
EXPLORATION 4.15 Making Sense of Wholes and Units 98
EXPLORATION 4.16 Multiplying Fractions 100
EXPLORATION 4.17 An Alternative Algorithm for Dividing Fractions 101
EXPLORATION 4.18 Remainders 102
EXPLORATION 4.19 Meanings of Operations with Fractions 103
EXPLORATION 4.20 Developing Operation Sense 104

4.4 • Exploring Beyond Integers and Fractions: Decimals, Exponents, and Real Numbers 106

EXPLORATION 4.21 Decimals and Base Ten Blocks 106
EXPLORATION 4.22 Exploring Decimal Algorithms 107

EXPLORATION 4.23 Patterns in Repeating Decimals 108
EXPLORATION 4.24 Developing Decimal Sense 110
EXPLORATION 4.25 The Right Bucket: A Decimal Game 112
EXPLORATION 4.26 Target: A Decimal Game 115

5 Proportional Reasoning 119

5.1 • Exploring Ratio and Proportion 119

EXPLORATION 5.1 Which Ramp Is Steeper? 120
EXPLORATION 5.2 Using Qualitative Reasoning to Develop Proportional Reasoning 121
EXPLORATION 5.3 Using Proportional Reasoning to Interpret Data 122
EXPLORATION 5.4 Unit Pricing and Buying Generic 125

5.2 • Exploring Percents 126

EXPLORATION 5.5 Percents 126
EXPLORATION 5.6 Do You Get What You Pay For? 128
EXPLORATION 5.7 Reducing, Enlarging, and Percents 129
EXPLORATION 5.8 Mice on Two Islands 131

6 Algebraic Thinking 133

6.1 • Exploring Patterns, Relations, and Functions 133

EXPLORATION 6.1 Growth Patterns 134
EXPLORATION 6.2 Guess the Function 139

6.2 • Exploring Mathematical Situations and Structures Using Algebraic Symbols 140

EXPLORATION 6.3 Relationships Between Variables 140
EXPLORATION 6.4 Exploring Equivalence 142
EXPLORATION 6.5 Exploring Inequalities 144

6.3 • Exploring Mathematical Models 145

EXPLORATION 6.6 Translating the Real World to Algebra 145
EXPLORATION 6.7 Proportional Reasoning and Functions 145

6.4 • Exploring Change in Various Contexts 147

EXPLORATION 6.8 Connecting Graphs and Words 147
EXPLORATION 6.9 More Pattern Block Explorations 150

7 Uncertainty: Data and Chance 151

7.1 • The Process of Collecting and Analyzing Data 151

EXPLORATION 7.1 Population Growth and Density 152
EXPLORATION 7.2 Collecting Data to Understand a Population: Typical Person 153
EXPLORATION 7.3 Exploring the Concept of Average 154
EXPLORATION 7.4 Explorations for Gathering and Analyzing Data 158

7.2 • Going Beyond the Basics 160

EXPLORATION 7.5 How Many Drops of Water Will a Penny Hold? 160
EXPLORATION 7.6 How Accurately Can You Drop the Whirlybird? 161

EXPLORATION 7.7 Exploring Relationships Among Body Ratios 162
EXPLORATION 7.8 Collecting Data to Make a Decision: What Container Is Best for Keeping Coffee Hot? 164
EXPLORATION 7.9 Explorations for Comparing Two Sets of Data 165
EXPLORATION 7.10 Collecting Data of Your Choice 168
EXPLORATION 7.11 Designing and Conducting a Survey 169

7.3 • Exploring Concepts Related to Chance 170

EXPLORATION 7.12 Heads and Tails and Probability 170
EXPLORATION 7.13 What Is the Probability of Having the Same Number of Boys and Girls? 171
EXPLORATION 7.14 What Is the Probability of Rolling Three Doubles in a Row? 172
EXPLORATION 7.15 What's in the Bag? 173
EXPLORATION 7.16 How Many Boxes Will You Probably Have to Buy? 175
EXPLORATION 7.17 More Simulations 176
EXPLORATION 7.18 Using Sampling to Estimate a Whole Population 177
EXPLORATION 7.19 Fair Games 178

7.4 • Exploring Counting and Chance 180

EXPLORATION 7.20 License Plates 180
EXPLORATION 7.21 Native American Games 181

8 **Geometry as Shape 185**

EXPLORATION 8.1 Geoboard Explorations 185
EXPLORATION 8.2 Tangram Explorations 191
EXPLORATION 8.3 Polyomino Explorations 197

8.1 • Exploring Basic Concepts of Geometry 203

EXPLORATION 8.4 Manhole Covers 203
EXPLORATION 8.5 Proof 207
EXPLORATION 8.6 Using Geometric Knowledge to Make Angles 208

8.2 • Exploring Two-Dimensional Figures 209

EXPLORATION 8.7 "What Do You See?" and "Make It from Memory" 209
EXPLORATION 8.8 Making Shapes from Folding a Square 213
EXPLORATION 8.9 Definitions and Language 214
EXPLORATION 8.10 The Sum of the Angles in a Polygon 216
EXPLORATION 8.11 Congruence 217
EXPLORATION 8.12 Polygons with Various Attributes 219
EXPLORATION 8.13 Polygons and Relationships 221

8.3 • Exploring Three-Dimensional Figures 225

EXPLORATION 8.14 Exploring Polyhedra 225
EXPLORATION 8.15 Relationships Among Polyhedra 227
EXPLORATION 8.16 Regular Polyhedra 228
EXPLORATION 8.17 Block Buildings 229
EXPLORATION 8.18 Cross Sections 233
EXPLORATION 8.19 Nets 234
EXPLORATION 8.20 Geometry Scavenger Hunt 240

9 Geometry as Measurement 241

9.1 • Exploring Systems of Measurement 241

EXPLORATION 9.1 How Far Is It? 241
EXPLORATION 9.2 How Tall? 243
EXPLORATION 9.3 How Thick? 244
EXPLORATION 9.4 How Much Is a Million? 245

9.2 • Exploring Perimeter and Area 246

EXPLORATION 9.5 What Does π Mean? 246
EXPLORATION 9.6 Exploring the Meaning of Area 247
EXPLORATION 9.7 Exploring Area on Geoboards 248
EXPLORATION 9.8 Exploring the Area of a Circle 253
EXPLORATION 9.9 Can You Make the Quilt Pattern? 254
EXPLORATION 9.10 How Much Will the Carpet Cost? 255
EXPLORATION 9.11 Irregular Areas 256
EXPLORATION 9.12 Exploring Relationships Between Perimeter and Area 257
EXPLORATION 9.13 Functions, Geometric Figures, and Geoboards 259

9.3 • Exploring Surface Area and Volume 265

EXPLORATION 9.14 Understanding Surface Area 265
EXPLORATION 9.15 Understanding Volume 266
EXPLORATION 9.16 Determining Volumes of Irregularly Shaped Objects 268
EXPLORATION 9.17 Paper Towels 269
EXPLORATION 9.18 Measurement, Ambiguity, and Precision 270
EXPLORATION 9.19 Applying Volume Concepts 271

10 Geometry as Transforming Shapes 273

EXPLORATION 10.1 Geoboard Explorations 273
EXPLORATION 10.2 Tangram Explorations 283
EXPLORATION 10.3 Polyomino Explorations 284

10.1 • Exploring Translations, Reflections, and Rotations 288

EXPLORATION 10.4 Reflections (Flips) 288
EXPLORATION 10.5 Paper Folding 295
EXPLORATION 10.6 Developing Rotation Sense 303

10.2 • Symmetry and Tessellations 307

EXPLORATION 10.7 Symmetries of Common Polygons 307
EXPLORATION 10.8 Symmetry Groups 308
EXPLORATION 10.9 Tessellations 313
EXPLORATION 10.10 Quilts 319

10.3 • Exploring Similarity 331

EXPLORATION 10.11 Similarity with Pattern Blocks 331
EXPLORATION 10.12 Similar Figures 333

Endnotes 335

Index 336

Cutouts

Base Ten Graph Paper
Other Base Graph Paper
Other Base Graph Paper
Other Base Graph Paper
Other Base Graph Paper
Geoboard Dot Paper
Isometric Dot Paper
Polyomino Grid Paper
Polyomino Grid Paper
Polyomino Grid Paper
Polyomino Grid Paper
Tangram Template
Regular Polygons
Exploring the Area of a Circle