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Representation Theory

A Homological Algebra Point of View

Introducing the representation theory of groups and finite dimensional algebras, this book first studies basic non-commutative ring theory, covering the necessary background of elementary homological algebra and representations of groups to block theory. It further discusses vertices, defect groups, Green and Brauer correspondences and Clifford theory. Whenever possible the statements are presented in a general setting for more general algebras, such as symmetric finite dimensional algebras over a field.

Then, abelian and derived categories are introduced in detail and are used to explain stable module categories, as well as derived categories and their main invariants and links between them. Group theoretical applications of these theories are given—such as the structure of blocks of cyclic defect groups—whenever appropriate. Overall, many methods from the representation theory of algebras are introduced.

Representation Theory assumes only the most basic knowledge of linear algebra, groups, rings and fields, and guides the reader in the use of categorical equivalences in the representation theory of groups and algebras. As the book is based on lectures, it will be accessible to any graduate student in algebra and can be used for self-study as well as for classroom use.

Mathematics

ISBN 978-3-319-07967-7



9783319079677

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1	Rings, Algebras and Modules	1
1.1	Basic Definitions	1
1.1.1	Algebras	1
1.1.2	Modules	3
1.2	Group Representations	7
1.2.1	Group Algebras and Their Modules	8
1.2.2	Maschke's Theorem	11
1.3	Noetherian and Artinian Objects	14
1.4	Wedderburn and Krull-Schmidt	16
1.4.1	The Krull-Schmidt Theorem	16
1.4.2	Wedderburn's Structure Theorem	22
1.4.3	Base Field Extension and Splitting Fields	31
1.5	Group Rings as Semisimple Algebras	33
1.6	Radicals and Jordan-Hölder Conditions	34
1.6.1	Jacobson- and Nil-Radical	35
1.6.2	The Jordan-Hölder Theorem	41
1.7	Tensor Products	48
1.7.1	The Definition and Elementary Properties	48
1.7.2	Immediate Applications for Group Rings	59
1.7.3	Mackey's Formula	66
1.8	Some First Steps in Homological Algebra; Extension Groups	72
1.8.1	Projective and Injective Modules	72
1.8.2	<i>Ext</i> -groups	80
1.8.3	Low Dimensional Group Cohomology	101
1.8.4	Short Exact Sequences of Groups and Group Cohomology	107
1.9	More on Projective Modules	116
1.9.1	The Module Approach	116
1.9.2	The Idempotent Approach	121
1.10	Injectives and Projectives May Coincide: Two Classes of Such Algebras	125
1.10.1	Self-Injective Algebras and Frobenius Algebras	125
1.10.2	Symmetric Algebras	134

1.11	Algebras Defined by Quivers and Relations:	
	Two Classes of Algebras	140
1.11.1	Hereditary Algebras	140
1.11.2	Special Biserial Algebras	145
	References	152
2	Modular Representations of Finite Groups	155
2.1	Relatively Projective Modules	155
2.1.1	Relatively Projective Modules for Subalgebras	155
2.1.2	Vertex and Source	161
2.1.3	Green Correspondence	165
2.2	Clifford Theory	169
2.2.1	Clifford's Main Theorem	169
2.2.2	Group Graded Rings and Green's Indecomposability Theorem	173
2.3	Brauer Correspondence	176
2.4	Properties of Defect Groups, Brauer and Green Correspondence	181
2.5	Orders and Lattices	185
2.5.1	Discrete Valuation Rings	185
2.5.2	Classical Orders and Their Lattices	189
2.6	The Cartan-Brauer Triangle	197
2.6.1	Grothendieck Groups	198
2.6.2	Cartan and Decomposition Maps	199
2.6.3	The Cartan-Brauer Triangle	202
2.7	Defining Blocks by Non-vanishing Ext^1 Between Simple Modules	204
2.8	The Structure of Serial Symmetric Algebras	209
2.9	Külshammer Ideals	214
2.9.1	Definitions of Külshammer Spaces	214
2.9.2	The Number of Simple Modules	217
2.9.3	Külshammer Ideals of Symmetric Algebras	219
2.9.4	Further Properties of Külshammer Ideals of Group Algebras	224
2.10	Brauer Constructions and p -Subgroups	228
2.10.1	The Brauer Homomorphism and Osima's Theorem	228
2.10.2	Higman's Criterion, Traces and the Brauer Quotient	231
2.10.3	Brauer Pairs	234
2.11	Blocks, Subgroups and Clifford Theory of Blocks	240
2.11.1	Brauer Correspondence for Bigger Groups	240
2.11.2	Blocks and Normal Subgroups	244

2.12	Representation Type of Blocks, Cyclic Defect.	251
2.12.1	The Structure of Blocks with Normal Cyclic Defect Group	251
2.12.2	Blocks of Finite Representation Type.	252
	References	256
3	Abelian and Triangulated Categories	259
3.1	Definitions	259
3.1.1	Categories.	259
3.1.2	Functors	263
3.1.3	Natural Transformations	266
3.2	Adjoint Functors	273
3.3	Additive and Abelian Categories	279
3.3.1	The Basic Definition	279
3.3.2	Useful Tools in Abelian Categories	281
3.4	Triangulated Categories	288
3.5	Complexes	299
3.5.1	The Category of Complexes	300
3.5.2	Homotopy Category of Complexes.	304
3.5.3	The Derived Category of Complexes	319
3.6	An Application and a Useful Tool: Hochschild Resolutions . . .	340
3.6.1	The Bar Resolution	340
3.6.2	Second Hochschild Cohomology, Extensions and The Wedderburn-Malcev Theorem.	343
3.6.3	Unicity of the Form on Frobenius Algebras	348
3.7	Hypercohomology and Derived Functors	349
3.7.1	Left and Right Derived Functors	350
3.7.2	Derived Tensor Product, Derived Hom Space, Hyper Tor and Hyper Ext.	352
3.8	Applications: Flat and Related Objects	360
3.9	Spectral Sequences.	371
3.9.1	Spectral Sequence of a Filtered Differential Module	372
3.9.2	Spectral Sequences of Filtered Complexes	376
3.9.3	Spectral Sequence of a Double Complex	378
3.9.4	Short Exact Sequences Associated to Spectral Sequences.	381
3.9.5	Application: Homomorphisms in the Derived Category	382
	References	384
4	Morita Theory.	387
4.1	Progenerators.	387
4.2	The Morita Theorem	393

4.3	Properties Invariant Under Morita Equivalences	400
4.4	Group Theoretical Examples	402
4.4.1	Elementary Examples	402
4.4.2	Nilpotent Blocks	406
4.5	Some Basic Algebras	412
4.5.1	Basic Hereditary Algebras and Gabriel's Theorem	412
4.5.2	Frobenius Algebras and Self-injective Algebras II	416
4.6	Picard Groups	419
	References	424
5	Stable Module Categories	427
5.1	Definitions, Elementary Properties, Examples	427
5.1.1	Definition of Stable Categories	427
5.1.2	Syzygies	428
5.1.3	Representation Type and Equivalences of Stable Categories	430
5.1.4	The Case of Self-Injective Algebras; Triangulated Structure	431
5.2	Some Examples of Stable Categories	440
5.3	Stable Equivalences of Morita Type: The Definition	443
5.3.1	Some Auxiliary Results on Functors Between Stable Categories	443
5.3.2	Broué's Original Definition	446
5.3.3	Stable Equivalences Induced by Exact Functors	448
5.4	Stable Equivalences of Morita Type Between Indecomposable Algebras	455
5.5	Stable Equivalences of Morita Type: Symmetricity and Further Properties	463
5.6	Image of Simple Modules Under Stable Equivalences of Morita Type	466
5.7	The Stable Grothendieck Group	469
5.7.1	The Definition	469
5.7.2	Application to Nakayama Algebras	472
5.8	Stable Equivalences and Hochschild (Co-)Homology	474
5.8.1	Stable Equivalence of Morita Type and Hochschild Homology Using Bouc's Trace	474
5.8.2	Stable Equivalences of Morita Type and Hochschild Cohomology	485
5.9	Degree Zero Hochschild (Co-)Homology	486
5.9.1	Stable Centre	486
5.9.2	Stable Cocentre	497
5.9.3	Külshammer Ideals and Stable Equivalences of Morita Type	500

5.10	Brauer Tree Algebras and the Structure of Blocks with Cyclic Defect Groups	504
5.10.1	Brauer Tree Algebras	505
5.10.2	Representation Finite Local Blocks	513
5.10.3	Algebras Which Are Stably Equivalent to a Symmetric Serial Algebra.	515
5.10.4	Blocks with Cyclic Defect Group.	528
5.11	Stable Equivalences and Self-Injectivity	534
5.11.1	Functor Categories	535
5.11.2	The Proof of Reiten's Theorem	546
	References	554
6	Derived Equivalences	557
6.1	Tilting Complexes	557
6.1.1	Historical and Methodological Remarks	557
6.1.2	An Equivalence Gives a Tilting Complex	558
6.2	Some Background on Co-algebras	563
6.2.1	Co-algebras, Bi-algebras, Hopf-Algebras.	564
6.2.2	Derivations, Coderivations	568
6.3	Strong Homotopy Action	574
6.3.1	Strong Homotopy Actions for Complexes Without Self-Extensions	574
6.3.2	Morphisms Between Strong Homotopy Actions.	577
6.3.3	Constructing an Algebra Action Out of a Strong Homotopy Action	580
6.4	Constructing a Functor from a Tilting Complex; Keller's Theorem.	587
6.5	Rickard's Morita Theorem for Derived Equivalences	589
6.5.1	The Theorem and Its Proof	589
6.5.2	Derived Equivalences of Standard Type	594
6.6	The Case of Two-Term Complexes	596
6.6.1	The Two-Sided Tilting Complex of a Two-Term Tilting Complex.	597
6.6.2	Endomorphism Rings of Two-Term Tilting Complexes	600
6.7	First Properties of Derived Equivalences.	605
6.7.1	A First Non-trivial Example	605
6.7.2	Local Rings and Commutative Rings	608
6.7.3	Hochschild (Co-)Homology and Derived Equivalences	612
6.7.4	Symmetric Algebras	616

6.8	Grothendieck Groups and Cartan-Brauer Triangle	618
6.8.1	The Grothendieck Group of a Triangulated Category	619
6.8.2	Derived Equivalences and Cartan-Brauer Triangles . . .	622
6.9	Singularity and Stable Categories as Quotients of Derived Categories.	627
6.9.1	Definition and First Properties of the Singularity Category	627
6.9.2	Singularity Categories and Self-Injective Algebras; Link to the Stable Case	631
6.9.3	Homomorphisms in Singularity Categories as Limits.	637
6.9.4	Algebras with Radical Squared 0; an Example	639
6.10	Examples	645
6.10.1	Blocks of Cyclic Defect	645
6.10.2	Okuyama's Method	651
6.10.3	Derived Equivalences Defined by Simple Modules . . .	657
6.11	Broué's Abelian Defect Group Conjecture; Historical Remarks.	665
6.12	Picard Groups of Derived Module Categories	668
6.12.1	The General Definition.	668
6.12.2	Fröhlich's Localisation Sequence.	672
6.12.3	An Example: The Brauer Tree Algebras.	677
	References	692
Index		697