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'This book is a window into the new field of mathematics emerging from the convergence of category theory and homotopy theory. It widens and deepens the extension from category theory into quasi-categories with the addition of a new theory of presheaves inspired by type theory and a new theory of localisation; it further proposes an extension of homotopical algebra to quasi-categories, offers new applications, and brings important simplification to earlier works. It is an excellent introduction to the subject and may be used for an advanced course.'

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CAMBRIDGE
UNIVERSITY PRESS
www.cambridge.org

ISBN 978-1-108-47320-0



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