## **Contents**

	PRE	FACE	XV
1	ASPECTS OF MULTIVARIATE ANALYSIS		
	1.1	Introduction 1	
	1.2	Applications of Multivariate Techniques 3	
	1.3	The Organization of Data 5  Arrays, 5  Descriptive Statistics, 6	
		Graphical Techniques, 11	
	1.4	Data Displays and Pictorial Representations 19 Linking Multiple Two-Dimensional Scatter Plots, 20 Graphs of Growth Curves, 24 Stars, 26 Chernoff Faces, 27	
	1.5	Distance 30	
	1.6	Final Comments 37	
	1.0	Exercises 37	
		References 47	
2	MA7	TRIX ALGEBRA AND RANDOM VECTORS	49
	2.1	Introduction 49	
	2.2	Some Basics of Matrix and Vector Algebra 49 Vectors, 49 Matrices, 54	
	2.3	Positive Definite Matrices 60	
	2.4	A Square-Root Matrix 65	
	2.5	Random Vectors and Matrices 66	
	2.6	Mean Vectors and Covariance Matrices 68 Partitioning the Covariance Matrix, 73 The Mean Vector and Covariance Matrix for Linear Combinations of Random Variables, 75 Partitioning the Sample Mean Vector and Covariance Matrix, 77	
	2.7	Matrix Inequalities and Maximization 78	

		Vectors, 82 Matrices, 87
		Exercises 103
		References 110
3	SAMI	PLE GEOMETRY AND RANDOM SAMPLING 111
	3.1	Introduction 111
	3.2	The Geometry of the Sample 111
	3.3	Random Samples and the Expected Values of the Sample Mean and Covariance Matrix 119
	3.4	Generalized Variance 123  Situations in which the Generalized Sample Variance Is Zero, 129  Generalized Variance Determined by \R    and Its Geometrical Interpretation, 134  Another Generalization of Variance, 137
	3.5	Sample Mean, Covariance, and Correlation As Matrix Operations 137
	3.6	Sample Values of Linear Combinations of Variables 140
		Exercises 144
		References 148
	4.1 4.2	Introduction 149  The Multivariate Normal Density and Its Properties 149  Additional Properties of the Multivariate  Normal Distribution, 156
	4.3	Normai Distribution, 130
		Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of η and Σ, 170  Sufficient Statistics, 173
	4.4	Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of η and Σ, 170
		Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of $\eta$ and $\Sigma$ , 170  Sufficient Statistics, 173  The Sampling Distribution of X and S 173
	4.4	Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of η and Σ, 170  Sufficient Statistics, 173  The Sampling Distribution of X and S 173  Properties of the Wishart Distribution, 174  Large-Sample Behavior of X and S 175  Assessing the Assumption of Normality 111  Evaluating the Normality of the Univariate Marginal Distributions, 177
	4.4	Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of $\eta$ and $\Sigma$ , 170  Sufficient Statistics, 173  The Sampling Distribution of X and S 173  Properties of the Wishart Distribution, 174  Large-Sample Behavior of X and S 175  Assessing the Assumption of Normality 111
	4.4 4.5 4.6	Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of η and Σ, 170  Sufficient Statistics, 173  The Sampling Distribution of X and S 173  Properties of the Wishart Distribution, 174  Large-Sample Behavior of X and S 175  Assessing the Assumption of Normality 111  Evaluating the Normality of the Univariate Marginal Distributions, 177  Evaluating Bivariate Normality, 182  Detecting Outliers and Cleaning Data 187  Steps for Detecting Outliers, 189  Transformations to Near Normality 192  Transforming Multivariate Observations, 195  Exercises 200
	4.4 4.5 4.6 4.7	Sampling from a Multivariate Normal Distribution and Maximum Likelihood Estimation 168  The Multivariate Normal Likelihood, 168  Maximum Likelihood Estimation of η and Σ, 170  Sufficient Statistics, 173  The Sampling Distribution of X and S 173  Properties of the Wishart Distribution, 174  Large-Sample Behavior of X and S 175  Assessing the Assumption of Normality 111  Evaluating the Normality of the Univariate Marginal Distributions, 17  Evaluating Bivariate Normality, 182  Detecting Outliers and Cleaning Data 187  Steps for Detecting Outliers, 189  Transformations to Near Normality 192  Transforming Multivariate Observations, 195

Supplement 2A: Vectors and Matrices: Basic Concepts 82

5	INFERENCES	$\Lambda D \cap I \mid T \mid \Lambda$	$\Lambda \Lambda \Box \Lambda \Lambda I$	V = C + C + C + C + C + C + C + C + C + C

210

- 5.1 Introduction 210
- 5.2 The Plausibility of  $\mu_{()}$  as a Value for a Normal Population Mean 210
- 5.3 Hotelling's T<sup>2</sup> and Likelihood Ratio Tests 216 General Likelihood Ratio Method, 219
- 5.4 Confidence Regions and Simultaneous Comparisons of Component Means 220
  Simultaneous Confidence Statements, 223
  A Comparison of Simultaneous Confidence Intervals with One-at-a-Time Intervals, 229
  The Bonferroni Method of Multiple Comparisons, 232
- 5.5 Large Sample Inferences about a Population Mean Vector 234
- Multivariate Quality Control Charts 239
   Charts for Monitoring a Sample of Individual Multivariate Observations for Stability, 241
   Control Regions for Future Individual Observations, 247
   Control Ellipse for Future Observations, 248
   T<sup>2</sup>-Chart for Future Observations, 248
   Control Charts Based on Subsample Means, 249
   Control Regions for Future Subsample Observations, 251
- 5.7 Inferences about Mean Vectors when Some Observations Are Missing 251
- 5.8 Difficulties Due to Time Dependence
   in Multivariate Observations 256
   Supplement 5A: Simultaneous Confidence Intervals and Ellipses
   as Shadows of the p-Dimensional Ellipsoids 258

Exercises 261 References 272

## 6 COMPARISONS OF SEVERAL MULTIVARIATE MEANS

273

- 6.1 Introduction 273
- Paired Comparisons and a Repeated Measures Design 273
   Paired Comparisons, 273
   A Repeated Measures Design for Comparing Treatments, 279
- 6.3 Comparing Mean Vectors from Two Populations 284

  Assumptions Concerning the Structure of the Data, 284

  Further Assumptions When n1 and n<sub>2</sub>Are Small, 285

  Simultaneous Confidence Intervals, 288

  The Two-Sample Situation When Σ1 # Σ2,291

  An Approximation to the Distribution of T<sup>2</sup> for Normal Populations

  When Sample Sizes Are Not Large, 294
- 6.4 Comparing Several Multivariate Population Means
   (One-Way Manova) 296
   Assumptions about the Structure of the Data for One-Way MANOVA, 296

		Multivariate Analysis of Variance (MANOVA), 301	
	6.5	Simultaneous Confidence Intervals for Treatment Effects 308	
	6.6	Testing for Equality of Covariance Matrices 310	
	6.7	Two-Way Multivariate Analysis of Variance 312	
		Univariate Two-Way Fixed-Effects Model with Interaction, 312	
		Multivariate Two-Way Fixed-Effects Model with Interaction, 315	
	6.8	Profile Analysis 323	
	6.9	Repeated Measures Designs and Growth Curves 328	
	6.10	Perspectives and a Strategy for Analyzing	
		Multivariate Models 332	
		Exercises 337	
		References 358	
7	MULT	IVARIATE LINEAR REGRESSION MODELS	360
	7.1	Introduction 360	
	7.2	The Classical Linear Regression Model 360	
	7.3	Least Squares Estimation 364 Sum-of-Squares Decomposition, 366	
		Geometry of Least Squares, 367	
		Sampling Properties of Classical Least Squares Estimators, 369	
	7.4	Inferences About the Regression Model 370	
		Inferences Concerning the Regression Parameters, 370	
		Likelihood Ratio Tests for the Regression Parameters, 374	
	7.5	Inferences from the Estimated Regression Function 378	
		Estimating the Regression Function at z <sub>()</sub> , 378	
	7.6	Forecasting a New Observation at $z_{()}$ , 379	
	7.6	Model Checking and Other Aspects of Regression 381	
		Does the Model Fit?, 381 Leverage and Influence, 384	
		Additional Problems in Linear Regression, 384	
	7.7	Multivariate Multiple Regression 387	
		Likelihood Ratio Tests for Regression Parameters, 395	
		Other Multivariate Test Statistics, 398	
		Predictions from Multivariate Multiple Regressions, 399	
	7.8	The Concept of Linear Regression 401	
		Prediction of Several Variables, 406	
	7.0	Partial Correlation Coefficient, 409	
	7.9	Comparing the Two Formulations of the Regression Model 410  Mean Corrected Form of the Regression Model, 410	
		Relating the Formulations, 412	
	7.10	Multiple Regression Models with Time Dependent Errors 413	
	7.10	Supplement 7A:The Distribution of the Likelihood Ratio	
		for the Multivariate Multiple Regression Model	418
		Exercises 420	410
		Pafarances 420	

A Summary of Univariate ANOVA, 297

- 8.1 Introduction 430
  8.2 Population Principal Components 430

  Principal Components Obtained from Standardized Variables, 436

  Principal Components for Covariance Matrices

  with Special Structures, 439
  8.3 Summarizing Sample Variation by Principal Components 441

  The Number of Principal Components 444
- 8.3 Summarizing Sample Variation by Principal Components 44

  The Number of Principal Components, 444

  Interpretation of the Sample Principal Components, 448

  Standardizing the Sample Principal Components, 449
- 8.4 Graphing the Principal Components 454
- 8.5 Large Sample Inferences 456

  Large Sample Properties of λ; and ei, 456

  Testing for the Equal Correlation Structure, 457
- 8.6 Monitoring Quality with Principal Components 459 Checking a Given Set of Measurements for Stability, 459 Controlling Future Values, 463

Supplement 8A: The Geometry of the Sample Principal Component Approximation 466

The p-Dimensional Geometrical Interpretation, 468 The n-Dimensional Geometrical Interpretation, 469

Exercises 470
References 480

## FACTOR ANALYSIS AND INFERENCE FOR STRUCTURED COVARIANCE MATRICES

481

- 9.1 Introduction 481
- 9.2 The Orthogonal Factor Model 482
- 9.3 Methods of Estimation 488

  The Principal Component (and Principal Factor) Method, 488

  A Modified Approach—the Principal Factor Solution, 494

  The Maximum Likelihood Method, 495

  A Large Sample Test for the Number of Common Factors, 501
- 9.4 Factor Rotation 504 *Oblique Rotations*; 512
- 9.5 Factor Scores 513
  The Weighted Least Squares Method, 514
  The Regression Method, 516
- 9.6 Perspectives and a Strategy for Factor Analysis 519
  Supplement 9A: Some Computational Details
  for Maximum Likelihood Estimation 527
  Recommended Computational Scheme, 528
  Maximum Likelihood Estimators of p = L<sub>2</sub>L'<sub>Z</sub> + \psi z 529
  Exercises 530

References 538

CANC	DNICAL CORRELATION ANALYSIS	539
10.1	Introduction 539	
10.2	Canonical Variates and Canonical Correlations 539	
10.3	Interpreting the Population Canonical Variables 545 Identifying the Canonical Variables, 545 Canonical Correlations as Generalizations of Other Correlation Coefficients, 547 The First r Canonical Variables as a Summary of Variability, 548 A Geometrical Interpretation of the Population Canonical Correlation Analysis 549	
10.4	The Sample Canonical Variates and Sample Canonical Correlations 550	
10.5	Additional Sample Descriptive Measures 558  Matrices of Errors of Approximations, 558  Proportions of Explained Sample Variance, 561	
10.6	Large Sample Inferences 563	
	Exercises 567	
	References 574	
DISCI	RIMINATION AND CLASSIFICATION	575
11.1	Introduction 575	
11.2	Separation and Classification for Two Populations 576	
11.3	Classification with Two Multivariate Normal Populations 584  Classification of Normal Populations When $\Sigma 1 = \Sigma_2 = \Sigma$ , 584  Scaling, 589  Fisher's Approach to Classification with Two Populations, 590  Is Classification a Good Idea?, 592  Classification of Normal Populations When $\Sigma 1 \# \Sigma_2$ ,593	
11.4	Evaluating Classification Functions 596	
11.5	Classification with Several Populations 606 The Minimum Expected Cost of Misclassification Method, 606 Classification with Normal Populations, 609	
11.6	Fisher's Method for Discriminating among Several Populations 621 Using Fisher's Discriminants to Classify Objects, 628	
11.7	Logistic Regression and Classification 634 Introduction, 634 The Logit Model, 634 Logistic Regression Analysis, 636 Classification, 638 Logistic Regression with Binomial Responses, 640	
11.8	Final Comments 644 Including Qualitative Variables, 644 Classification Trees, 644 Neural Networks, 647 Selection of Variables, 648	

		Testing for Group Differences, 648 Graphics, 649 Practical Considerations Regarding Multivariate Normality, 649 Exercises 650 References 669	
12	CLUS	TERING. DISTANCE METHODS. AND ORDINATION	671
	12.1	Introduction 671	
	12.2	Similarity Measures 673 Distances and Similarity Coefficients for Pairs of Items, 673 Similarities and Association Measures for Pairs of Variables, 677 Concluding Comments on Similarity, 678	
	12.3	Hierarchical Clustering Methods 680 Single Linkage, 682 Complete Linkage, 685 Average Linkage, 690 Ward's Hierarchical Clustering Method, 692 Final Comments—Hierarchical Procedures, 695	
	12.4	Nonhierarchical Clustering Methods 696 K-means Method, 696 Final Cqmments—Nonhierarchical Procedures, 701	
	12.5	Clustering Based on Statistical Models 703	
	12.6	Multidimensional Scaling 706 The Basic Algorithm, 708	
	12.1	Correspondence Analysis 716 Algebraic Development of Correspondence Analysis, 718 Inertia, 725 Interpretation in Two Dimensions, 726 Final Comments, 726	
	12.8	Biplots for Viewing Sampling Units and Variables 726  Constructing Biplots, 727	
	12.9	Procrustes Analysis: A Method for Comparing Configurations 732  Constructing the Procrustes Measure of Agreement, 733  Supplement 12A: Data Mining 740  Introduction, 740  The Data Mining Process, 741  Model Assessment, 742  Exercises 747  References 755	
	ADDI	ENDIX	757
		A INDEX	764
	SUB	JECT INDEX	767