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Faced with questions like "What is justice?", practical people discuss practical matters, and intellectuals enjoy the vague contemplation of more abstract topics. The basic philosophers provide answers which may not be found, but which illuminate the path that we follow.

The question "What are numbers?" is clearly less important, but has interested several philosophers and mathematicians. The answer given in this book is essentially due to Cantor, Dedekind in two essays ("Stetigkeit und irrationale Zahlen" (continuity and irrational numbers)) and "Was sind und was sollen die Zahlen?" (What are numbers and what should they be? [1]). Starting with the natural numbers,  $\mathbb{N}$  (that is to say, the strictly positive integers), we consider the strictly positive rational numbers  $\mathbb{Q}^+$  and then use these to construct the rational numbers  $\mathbb{Q}$ . We then use the rational numbers to construct the real numbers. But once  $\mathbb{Q}$  has been constructed, we construct the complex numbers  $\mathbb{C}$  and we consider the numbers required by modern analysts (that is to say, Dedekind cuts) to complete the construction of  $\mathbb{C}$ . (This is the reason why we have to start with the natural numbers.) However, we will have to say "when?" comes from. Dedekind showed that all the properties of the strictly positive integers can be derived from a very small number of very plausible rules. The question of whether to accept these rules cannot be left to the individual mathematician.

In the real world, we dig the foundations before we start the building. In university mathematics we have tended to install foundations when the building is almost complete. Pedagogically, there are good reasons for studying the construction of the various number systems only after the student has acquired facility in using them.