

# Contents

<b>Introduction</b>		<b>1</b>
<b>PART I THE RATIONALS</b>		<b>5</b>
<b>1 Counting Sheep</b>		<b>7</b>
1.1 A Foundation Myth		7
1.2 What Were Numbers Used For?		12
1.3 A Greek Myth		15
<b>2 The Strictly Positive Rationals</b>		<b>23</b>
2.1 An Indian Legend		23
2.2 Equivalence Classes		27
2.3 Properties of the Strictly Positive Rationals		33
2.4 What Have We Actually Done?		37
<b>3 The Rational Numbers</b>		<b>39</b>
3.1 Negative Numbers		39
3.2 Defining the Rational Numbers		44
3.3 What Does Nature Say?		51
3.4 When Are Two Things the Same?		52
<b>PART II THE NATURAL NUMBERS</b>		<b>59</b>
<b>4 The Golden Key</b>		<b>61</b>
4.1 The Least Member		61
4.2 Inductive Definition		65
4.3 Applications		69
4.4 Prime Numbers		77
<b>5 Modular Arithmetic</b>		<b>83</b>
5.1 Finite Fields		83



5.2	Some Pretty Theorems	87
5.3	A New Use for Old Numbers	91
5.4	More Modular Arithmetic	98
5.5	Problems of Equal Difficulty	101
<b>6</b>	<b>Axioms for the Natural Numbers</b>	109
6.1	The Peano Axioms	109
6.2	Order	113
6.3	Conclusion of the Argument	117
6.4	Order Numbers Can Be Used as Counting Numbers	121
6.5	Objections	127
	<b>PART III THE REAL NUMBERS (AND THE COMPLEX NUMBERS)</b>	135
<b>7</b>	<b>What Is the Problem?</b>	137
7.1	Mathematics Becomes a Profession	137
7.2	Rogue Numbers	138
7.3	How Can We Justify Calculus?	147
7.4	The Fundamental Axiom of Analysis	151
7.5	Dependent Choice	156
7.6	Equivalent Forms of the Fundamental Axiom	159
<b>8</b>	<b>And What Is Its Solution?</b>	167
8.1	A Construction of the Real Numbers	167
8.2	Some Consequences	177
8.3	Are the Real Numbers Real?	182
<b>9</b>	<b>The Complex Numbers</b>	187
9.1	Constructing the Complex Numbers	187
9.2	Analysis for $\mathbb{C}$	191
9.3	Continuous Functions from $\mathbb{C}$	195
<b>10</b>	<b>A Plethora of Polynomials</b>	199
10.1	Preliminaries	199
10.2	The Fundamental Theorem of Algebra	205
10.3	Liouville Numbers	209
10.4	A Non-Archimedean Ordered Field	213
<b>11</b>	<b>Can We Go Further?</b>	221
11.1	The Quaternions	221
11.2	What Happened Next	226
11.3	Valedictory	230



APPENDICES	231
<i>Appendix A</i> <b>Products of Many Elements</b>	233
<i>Appendix B</i> <b><math>n</math>th Complex Roots</b>	239
<i>Appendix C</i> <b>How Do Quaternions Represent Rotations?</b>	243
<i>Appendix D</i> <b>Why Are the Quaternions So Special?</b>	247
<i>References</i>	255
<i>Index</i>	257