

# Contents

## Preface

|                                                  |    |
|--------------------------------------------------|----|
| 1 Probability tools for stochastic modelling     | 1  |
| 1 The sample space                               | 1  |
| 2 Probability space                              | 3  |
| 3 Random variables                               | 7  |
| 4 Integrability, Expectation and Independence    | 10 |
| 5 Main distribution probabilities                | 16 |
| 5.1 The binomial distribution                    | 16 |
| 5.2 The Poisson distribution                     | 17 |
| 5.3 The normal (or Laplace-Gauss) distribution   | 18 |
| 5.4 The log-normal distribution                  | 20 |
| 5.5 The negative exponential distribution        | 21 |
| 5.6 The multidimensional normal distribution     | 22 |
| 6 Conditioning (From independence to dependence) | 24 |
| 6.1 Conditioning: introductory case              | 24 |
| 6.2 Conditioning: general case                   | 28 |
| 6.3 Regular conditional probability              | 32 |
| 7 Stochastic processes                           | 36 |
| 8 Martingales                                    | 40 |
| 9 Brownian motion                                | 43 |
| 2 Renewal theory                                 | 45 |
| 1 Purpose                                        | 45 |
| 2 Main definitions                               | 46 |
| 3 Classification of renewal processes            | 47 |
| 4 The renewal equation                           | 52 |
| 5 The use of Laplace transform                   | 59 |
| 5.1 The Laplace transform                        | 59 |
| 5.2 The Laplace-Stieltjes transform              | 64 |
| 5.3 An application to the renewal function       | 66 |
| 6 Application of Wald's identity                 | 67 |
| 6.1 Wald's identity                              | 67 |
| 6.2 A lower bound for renewal function $R$       | 68 |
| 7 Asymptotical behaviour of the $N(t)$ -process  | 69 |
| 8 Recurrence times                               | 72 |
| 8.1 Definitions                                  | 72 |
| 8.2 Distributions of recurrence times            | 73 |
| 8.3 Asymptotic behaviour                         | 79 |

|                                                                                     |            |
|-------------------------------------------------------------------------------------|------------|
| 8.4 Example: the Poisson process                                                    | 82         |
| 8.5 Some parameters of the limit distributions                                      | 83         |
| 8.6 A characterization of the Poisson process                                       | 86         |
| 9 Delayed and stationary renewal processes                                          | 87         |
| 10 Numerical aspects                                                                | 92         |
| 10.1 General quadrature method                                                      | 93         |
| 10.2 Some particular formulas                                                       | 95         |
| 10.3 Numerical approximation of the Poisson process                                 | 97         |
| 10.4 A real life example in motor-car accidents                                     | 98         |
| 10.4.1 Data description                                                             | 98         |
| 10.4.2 The result distribution                                                      | 101        |
| <b>3 Markov chains</b>                                                              | <b>105</b> |
| 1 Definitions                                                                       | 105        |
| 2 Markov chain state classification                                                 | 108        |
| 2.1 Periodic and aperiodic states                                                   | 108        |
| 2.2 Essential and inessential states – Irreducibility                               | 109        |
| 2.3 Transient and recurrent states                                                  | 110        |
| 3 Occupation times                                                                  | 113        |
| 4 Computation of absorption probabilities                                           | 113        |
| 5 Asymptotic behaviour                                                              | 115        |
| 6 Examples                                                                          | 119        |
| 7 A case study in a social insurance problem (Janssen (1966))                       | 122        |
| 8 Asymptotic numerical treatment                                                    | 125        |
| 8.1 An algorithm for MC Asymptotical Study                                          | 125        |
| 8.2 Irreducible real data example in motor-car insurance                            | 129        |
| 8.3 Reducible and uni-irreducible examples-Canonical form connection                | 133        |
| <b>4 Markov renewal theory, Markov random walks and semi-Markov processes.</b>      | <b>145</b> |
| 1 Positive (J-X) processes                                                          | 145        |
| 2 Semi-Markov and extended semi-Markov chains                                       | 145        |
| 3 Primary properties                                                                | 147        |
| 4 Examples                                                                          | 151        |
| 5 Markov renewal processes, semi-Markov processes and associated counting processes | 154        |
| 6 Markov renewal functions                                                          | 156        |
| 7 Classification of the state of a MRP                                              | 161        |
| 8 The Markov renewal equation                                                       | 163        |
| 9 Asymptotic behaviour of a MRP                                                     | 165        |
| 9.1 Asymptotic behaviour of Markov renewal functions                                | 165        |
| 9.2 Asymptotic behaviour of solutions of Markov renewal                             |            |

|                                                                       |         |
|-----------------------------------------------------------------------|---------|
| equations                                                             | 168     |
| 10 Asymptotic behaviour of SMP                                        | 169     |
| 10.1 Irreducible case                                                 | 169     |
| 10.2 Non-irreducible case                                             | 171     |
| 10.2.1 Uni-reducible case                                             | 172     |
| 10.2.2 General case                                                   | 173     |
| 11 Recurrence times                                                   | 173     |
| 11.1 Definitions                                                      | 173     |
| 11.2 Transient results                                                | 174     |
| 11.3 Asymptotic results                                               | 175     |
| 12 Delayed and stationary MRP                                         | 178     |
| 13 Particular cases of MRP                                            | 185     |
| 13.1 Renewal processes and Markov chains                              | 185     |
| 13.2 SMP an MRP of zero order (Pyke (1962))                           | 185     |
| 13.2.1 First type of zero order MRP                                   | 186     |
| 13.2.2 Second type of zero order MRP                                  | 186     |
| 13.3 Continuous Markov processes                                      | 187     |
| 14 A case study in social insurance (Janssen (1966))                  | 187     |
| 14.1 The semi-Markov model                                            | 187     |
| 14.2 Comparison of Markov and semi-Markov models                      | 188     |
| 15 Asymptotical numerical examples                                    | 189     |
| <br>5 Functionals of (J-X) processes                                  | <br>193 |
| 1 (J-X) processes                                                     | 193     |
| 2 Functionals of (J-X) processes                                      | 195     |
| 3 Functionals of positive (J-X) processes                             | 201     |
| 4 Classical random walks and risk theory                              | 202     |
| 4.1 Purpose                                                           | 202     |
| 4.2 Basic notions on random walks                                     | 202     |
| 4.3 Renewal results on ladder variables                               | 207     |
| 4.4 Classification of random walks                                    | 210     |
| 5 Defective positive (J-X) processes                                  | 214     |
| 6 Semi-Markov random walks                                            | 219     |
| 7 Indice variables for semi-Markov random walks                       | 222     |
| 8 Distribution of the supremum for semi-Markov random walks           | 224     |
| <br>6 Non-homogeneous Markov and semi-Markov processes.               | <br>227 |
| 1 General definitions                                                 | 227     |
| 1.1 Completely non homogeneous semi-Markov processes                  | 227     |
| 1.2 Special cases                                                     | 231     |
| 1.2.1 Non-homogeneous Markov additive process and semi-Markov process | 231     |

|                                                      |         |
|------------------------------------------------------|---------|
| 1.2.2 Non-homogeneous MC                             | 232     |
| 1.2.3 Homogeneous Markov additive process            | 232     |
| 1.2.4 Non-homogeneous renewal process                | 232     |
| 1.3 Intensities for CNHSMP                           | 233     |
| 1.3.1 Definition                                     | 233     |
| 1.3.2 Generalized Polya processes                    | 234     |
| 2 Non-homogeneous Markov chains                      | 235     |
| 3 Non-homogeneous Markov processes                   | 236     |
| 3.1 Classical definition                             | 237     |
| 3.2 The Chapman Kolmogorov equations                 | 237     |
| 3.3 Intensity functions                              | 238     |
| 3.4 The special case of constant intensity functions | 241     |
| 3.5 NHMP as a special case of NHSMP                  | 243     |
| 4 An NHMC transportation example                     | 243     |
| <br>7 Markov and semi-Markov reward processes.       | <br>247 |
| 1 Reward structure                                   | 247     |
| 1.1 Classification and notation                      | 247     |
| 1.2 Discrete time fixed interest rate                | 249     |
| 1.3 Discrete time variable interest rate             | 249     |
| 1.4 Continuous time fixed interest rate intensity    | 250     |
| 1.5 Continuous time variable interest rate intensity | 251     |
| 2 Discrete time Markov reward processes              | 252     |
| 2.1 Undiscounted case                                | 252     |
| 2.1.1 First model                                    | 252     |
| 2.1.2 Second model                                   | 253     |
| 2.1.3 Third model                                    | 253     |
| 2.1.4 Fourth model                                   | 254     |
| 2.1.5 Fifth model                                    | 254     |
| 2.1.6 Sixth model                                    | 255     |
| 2.1.7 Seventh model                                  | 255     |
| 2.1.8 Eighth model                                   | 256     |
| 2.2 Discounted case                                  | 256     |
| 2.2.1 Immediate cases                                | 256     |
| 2.2.2 Due cases                                      | 261     |
| 2.3 General algorithm for the DTMRWP                 | 264     |
| 2.3.1 Homogeneous MRWP                               | 264     |
| 2.3.2 Non-homogeneous MRWP                           | 265     |
| 2.4 A DTMRWP applied example                         | 265     |
| 3 Semi-Markov reward processes                       | 268     |
| 3.1 Undiscounted CTSMRWP                             | 268     |
| 3.1.1 First model                                    | 268     |
| 3.1.2 Second model                                   | 269     |

|                                     |     |
|-------------------------------------|-----|
| 3.1.3 Third model                   | 270 |
| 3.1.4 Fourth model                  | 271 |
| 3.1.5 Fifth model                   | 272 |
| 3.1.6 Sixth model                   | 272 |
| 3.1.7 Seventh model                 | 273 |
| 3.2 Discounted CTSMRWP              | 273 |
| 3.2.1 First model                   | 273 |
| 3.2.2 Second model                  | 275 |
| 3.2.3 Third model                   | 275 |
| 3.2.4 Fourth model                  | 276 |
| 3.2.5 Fifth model                   | 277 |
| 3.2.6 Sixth model                   | 277 |
| 3.2.7 Seventh model                 | 278 |
| 3.2.8 Eighth model                  | 278 |
| 3.2.9 Ninth model                   | 279 |
| 4 Insurance applications of CTSMRWP | 280 |
| 4.1 Two state examples              | 282 |
| 4.2 Three state examples            | 286 |
| 4.3 Four state examples             | 290 |
| References                          | 295 |
| Author index                        | 303 |
| Subject index                       | 305 |