

COMBINATORIAL MATHEMATICS

- Covers enumeration, graphs, sets, and methods
- Assumes no prior exposure to combinatorics
- Suitable for a one-year sequence, or as a one-semester introduction leading to advanced courses
- Contains more than 2200 exercises at various levels
- Complete solutions manual available for instructors at www.cambridge.org/west
- Includes more than 2300 references and many exercises, theorems, and proofs that have not previously appeared in textbooks
- Includes both classical results and more recent developments

"This is a real gem, capturing the spirit, breadth, and depth of combinatorics. Doug West is a master of exposition, his thorough treatment of the subject will be useful for students and researchers in mathematics and computer science."

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ALEXANDR KOSTOCHKA, University of Illinois

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Preface

xiii

Chapter 0 – Introduction.

1

Sets, Functions, and Relations: 2. Graphs: 3.

Discrete Probability: 6. Other Discrete Structures: 8. Complexity: 9.

Part I — Enumeration

Chapter 1 – Combinatorial Arguments.

13

1.1. Classical Models.

14

Elementary Principles: 14. Words, Sets, and Multisets: 16.

Exercises: 20.

1.2. Identities.

24

Lattice Paths and Pascal's Triangle: 24. Delannoy Numbers: 28.

Exercises: 31.

1.3. Applications.

35

Graphs and Trees: 35. Multinomial Coefficients: 37.

The Ballot Problem: 39. Catalan Numbers: 41. Exercises: 46.

Chapter 2 – Recurrence Relations.

51

2.1. Obtaining Recurrences.

52

Classical Examples: 52. Variations: 56. Exercises: 59.

2.2. Elementary Solution Methods.

66

The Characteristic Equation Method: 67.

The Generating Function Method: 73. Exercises: 78.

2.3. Further Topics.

82

The Substitution Method: 82. Asymptotic Analysis: 84.

The WZ Method (optional): 87. Exercises: 91.

Chapter 3 – Generating Functions.	93
3.1. Ordinary Generating Functions.	93
Modeling Counting Problems: 94. Permutation Statistics: 98.	
Exercises: 103.	
3.2. Coefficients and Applications.	107
Operations and Summations: 107. Snake Oil: 112. Exercises: 114.	
3.3. Exponential Generating Functions.	118
Modeling Labeled Structures: 118. Stirling and Derangement	
Applications: 121. The Exponential Formula: 124.	
The Lagrange Inversion Formula (optional): 129. Exercises: 132.	
3.4. Partitions of Integers.	137
Generating Function Methods: 137. Ferrers Diagrams: 141.	
Bulgarian Solitaire (optional): 143. Distribution Models: 145.	
Exercises: 148.	

Chapter 4 – Further Topics. 153

4.1. The Inclusion-Exclusion Principle.	153
The Basic Principle: 153. Restricted Permutations: 159.	
Signed Involutions: 163.	
Determinants and Path Systems (optional): 165. Exercises: 171.	
4.2. Pólya–Redfield Counting.	178
Burnside’s Lemma: 179. The Pattern Inventory: 181.	
Classical Cycle Indices: 185. Exercises: 187.	
4.3. Permutations and Tableaux.	189
The Hook-Length Formula: 189. The RSK Correspondence: 193.	
Switching P -Symbol and Q -Symbol: 198.	
Jeu de Taquin (optional): 201. Exercises: 206.	

Part II — Graphs

Chapter 5 – First Concepts for Graphs. 209

5.1. Definitions and Examples.	209
Graphs and Subgraphs: 209. Isomorphism: 211.	
The Petersen Graph and Hypercubes: 213. Exercises: 216.	
5.2. Vertex Degrees.	220
The Degree-Sum Formula: 220. Degree Lists: 221.	
Extremality: 223. Directed Graphs: 224. Exercises: 226.	
5.3. Connection and Decomposition.	229
Components and Walks: 229. Cycles and Cut-Edges: 231.	
Eulerian Circuits: 233. Exercises: 235.	
5.4. Trees and Distance.	239
Properties of Trees: 240. Distance and Diameter: 242.	
Optimization on Weighted Graphs: 245. Exercises: 247.	

Chapter 6 – Matchings.	253
6.1. Matching in Bipartite Graphs.	253
Hall's Theorem: 254. Min-Max Relations: 258. Exercises: 260.	
6.2. Matching in General Graphs.	264
Tutte's 1-Factor Theorem: 264. General Factors of Graphs: 268.	
Exercises: 272.	
6.3. Algorithmic Aspects.	276
Augmenting Paths: 277. Weighted Bipartite Matching: 279.	
Fast Bipartite Matching (optional): 283.	
Stable Matchings (optional): 284. Exercises: 286.	
Chapter 7 – Connectivity and Cycles.	289
7.1. Connectivity Parameters.	289
Separating Sets: 289. Edge Cuts: 293. Blocks: 294. Exercises: 295.	
7.2. Properties of k -Connected Graphs.	298
Menger's Theorem: 298. Applications of Menger's Theorem: 301.	
2-Connected and 3-Connected Graphs: 304.	
Highly Connected Orientations (optional): 307. Exercises: 311.	
7.3. Spanning Cycles.	316
Properties of Hamiltonian Graphs: 317.	
Sufficient Conditions: 319. Long Cycles (optional): 322.	
Further Directions (optional): 325. Exercises: 328.	
Chapter 8 – Coloring.	335
8.1. Vertex Coloring.	335
Upper Bounds: 336. Triangle-Free Graphs: 339. Exercises: 341.	
8.2. Structural Aspects.	344
Color-Critical Graphs: 344. List Coloring: 346.	
Forced Subgraphs (optional): 349. Exercises: 353.	
8.3. Edge-Coloring and Perfection.	357
Special Classes: 357. Vizing's Theorem and Extensions: 359.	
List Edge-Coloring: 362. Perfect Graphs: 366. Exercises: 370.	
Chapter 9 – Planar Graphs.	377
9.1. Embeddings and Euler's Formula.	377
Drawings and Duals: 378. Euler's Formula: 383. Exercises: 385.	
9.2. Structure of Planar Graphs.	390
Kuratowski's Theorem: 390. The Separator Theorem (optional): 394.	
Exercises: 397.	
9.3. Coloring of Planar Graphs.	399
Edge-Colorings and Spanning Cycles: 400.	
5-Colorable and 5-Choosable: 402. The Four Color Problem: 404.	
Discharging and Light Edges: 407.	
Other Aspects of Discharging (optional): 412. Exercises: 417.	

Part III — Sets

Chapter 10 – Ramsey Theory.	425
10.1. The Pigeonhole Principle.	425
Classical Applications: 426. Monotone Sublists: 430.	
Pattern-Avoiding Permutations (optional): 431.	
Large Girth and Chromatic Number (optional): 434.	
Edge-Coloring of Hypergraphs (optional): 438. Exercises: 440.	
10.2. Ramsey's Theorem.	443
The Main Theorem: 443. Applications: 445. Ramsey Numbers: 448.	
Graph Ramsey Theory: 453. Exercises: 458.	
10.3. Further Topics.	461
Van der Waerden's Theorem: 461. Infinite Sets (optional): 468.	
The Canonical Ramsey Theorem (optional): 470. Exercises: 473.	
Chapter 11 – Extremal Problems.	475
11.1. Forced Subgraphs.	475
Turán's Theorem: 475. Erdős–Stone Theorem: 478.	
Linear Ramsey for Bounded Degree: 483. Roth's Theorem: 484.	
Proof of the Regularity Lemma (optional): 486. Exercises: 489.	
11.2. Families of Sets.	493
The Kruskal–Katona Theorem: 493.	
Antichains and Intersecting Families: 496. Chvátal's Conjecture: 501.	
Sunflowers (optional): 503. Entropy (optional): 504. Exercises: 510.	
11.3. Matroids.	513
Hereditary Systems and Examples: 514. Axiomatics of Matroids: 519.	
Duality and Minors: 523. The Span Function: 527.	
Matroid Intersection: 529. Matroid Union: 533. Exercises: 536.	
Chapter 12 – Partially Ordered Sets.	541
12.1. Structure of Posets.	541
Definitions and Examples: 541. Dilworth's Theorem and Beyond: 546.	
Exercises: 549.	
12.2. Symmetric Chains and LYM Orders.	552
Graded Posets: 552. Symmetric Chain Decompositions: 553.	
LYM and Sperner Properties: 558.	
Products of LYM Orders (optional): 562. Exercises: 564.	
12.3. Linear Extensions and Dimension.	568
Order Dimension: 569. Computation and Bounds: 572.	
Bipartite Posets: 575. Exercises: 582.	
12.4. Special Families of Posets.	585
Semiororders and Interval Orders: 585. Lattices: 588.	
Distributive Lattices: 591. Correlational Inequalities: 595.	
A Problem in Ramsey Theory (optional): 601. Exercises: 604.	

Chapter 13 – Combinatorial Designs. 609

13.1. Arrangements. 609

Latin Squares: 609. Block Designs: 612. Symmetric Designs: 615.
Hadamard Matrices: 618. Exercises: 622.

13.2. Projective Planes. 624

Relation to Designs: 624. Applications to Extremal Problems: 628.
Difference Sets: 634. Exercises: 638.

13.3. Further Constructions. 640

Steiner Triple Systems: 641. Graphical Designs: 645.
Resolvable Designs and Other Tools: 648.
The Euler Conjecture (optional): 651. Exercises: 654.

Part IV — Methods

Chapter 14 – The Probabilistic Method. 657

14.1. Existence and Expectation. 657

The Union Bound: 658. Random Variables: 662. Exercises: 666.

14.2. Refinements of Basic Methods. 670

Deletions and Alterations: 670. The Symmetric Local Lemma: 674.
The General Local Lemma (optional): 679. Exercises: 684.

14.3. Moments and Thresholds. 686

“Almost Always”: 687. Threshold Functions: 690.
Convergence of Moments: 694. Graph Evolution: 698. Exercises: 702.

14.4. Concentration Inequalities. 706

Chebyshev and Chernoff Bounds: 706. Martingales: 712.
Bounded Differences (optional): 719. Exercises: 721.

Chapter 15 – Linear Algebra. 723

15.1. Dimension and Polynomials. 723

The Dimension Argument: 723. Restricted Intersections
of Sets (optional): 727. Combinatorial Nullstellensatz: 732.
The Alon–Tarsi Theorem: 738. Exercises: 744.

15.2. Matrices. 747

Determinants and Trees: 747. Cycle Space and Bond Space: 752.
Permanents and Planar Graphs: 754.
Möbius Inversion (optional): 757. Exercises: 762.

15.3. Eigenvalues. 766

Spectra of Graphs: 766. Eigenvalues and Graph Parameters: 768.
Regular and Strongly Regular Graphs: 772.
Laplacian Eigenvalues: 776. Exercises: 780.

Chapter 16 – Geometry and Topology.	783
16.1. Graph Drawings.	783
Embeddings on Grids: 783. Crossing Number: 790. Exercises: 797.	
16.2. Combinatorial Topology.	798
Sperner's Lemma and Bandwidth: 799. Equivalent Topological Lemmas: 802. The Borsuk–Ulam Theorem: 806. Kneser Conjecture and Gale's Lemma: 811. Ham Sandwiches and Bisections: 815. Borsuk's Conjecture: 817. Exercises: 818.	
16.3. Volumes and Containment.	821
Monotone Subsequences: 821. Balanced Comparisons: 822. Containment Orders: 828. Exercises: 831.	

Hints to Selected Exercises	833
------------------------------------	------------

References	849
-------------------	------------

Author Index	929
---------------------	------------

Glossary of Notation	943
-----------------------------	------------

Subject Index	949
----------------------	------------