Sources and Studies in the History of Mathematics and Physical Sciences

Jan von Plato

# Can Mathematics Be Proved Consistent?

## Gödel's Shorthand Notes & Lectures on Incompleteness

Kurt Gödel (1906–1978) shook the mathematical world in 1931 by a result that has become an icon of 20th century science: The search for rigour in proving mathematical theorems had led to the formalization of mathematical proofs, to the extent that such proving could be reduced to the application of a few mechanical rules. Gödel showed that whenever the part of mathematics under formalization contains elementary arithmetic, there will be arithmetical statements that should be formally provable but aren't. The result is known as Gödel's first incompleteness theorem, so called because there is a second incompleteness result, embodied in his answer to the question "Can mathematics be proved consistent?"

This book offers the first examination of Gödel's preserved notebooks from 1930, written in a long-forgotten German shorthand, that show his way to the results: his first ideas, how they evolved, and how the jewel-like final presentation in his famous publication *On formally undecidable propositions* was composed.The book also contains the original version of Gödel's incompleteness article, as handed in for publication with no mentioning of the second incompleteness theorem, as well as six contemporary lectures and seminars Gödel gave between 1931 and 1934 in Austria, Germany, and the United States. The lectures are masterpieces of accessible presentations of deep scientific results, readable even for those without special mathematical training, and published here for the first time.

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PART I: GÖDEL'S STEPS TOWARD INCOMPLETENESS ..... 1

1. The completeness problem ..... 3
2. From Skolem's paradox to the Königsberg conference ..... 8
3. From the Königsberg conference to von Neumann's letter ..... 13
4. The second theorem: "Only in a realm of ideas" ..... 24

PART II: THE SAVED SOURCES ON INCOMPLETENESS ..... 29

1. Shorthand writing ..... 31
2. Description of the shorthand notebooks on incompleteness ..... 33
3. The typewritten manuscripts ..... 46
4. Lectures and seminars on incompleteness ..... 49

PART III: THE SHORTHAND NOTEBOOKS ..... 59

1. Undecidability draft. We lay as a basis the system of the *Principia* ..... 61
2. There are unsolvable problems in the *Principia Mathematica* ..... 68
3. The development of mathematics in the direction of greater exactness ... 86
4. The question whether each mathematical problem is solvable ..... 102
5. A proof in broad outline will be sketched ..... 123
6. We produce an undecidable proposition in the *Principia* ..... 126
7. The development of mathematics in the direction of greater exactness .. 133
8. Let us turn back to the undecidable proposition ..... 162

PART IV: THE TYPEWRITTEN MANUSCRIPTS ..... 169

1. Some metamathematical results ..... 171
2. On formally undecidable propositions (earlier version) ..... 173

PART V: LECTURES AND SEMINARS ON INCOMPLETENESS ..... 201

1. Lecture on undecidable propositions (Bad Elster) ..... 203
2. On formally undecidable propositions (Bad Elster) ..... 206
3. On undecidable propositions (Vienna) ..... 213
4. On the impossibility of proofs of freedom from contradiction (Vienna) .. 226
5. The existence of undecidable propositions (New York) ..... 235
6. Can mathematics be proved consistent? (Washington) ..... 246

Index of names and list of references in Gödel's notes ..... 261

References for Parts I and II ..... 262