mathematics

TOPOLOGY A Categorical Approach

Tai-Danae Bradley, Tyler Bryson, and John Terilla

This graduate-level textbook on topology takes a unique approach: it reintroduces basic, point-set topology from a more modern, categorical perspective. Many graduate students are familiar with the ideas of point-set topology and they are ready to learn something new about them. Teaching the subject using category theory—a contemporary branch of mathematics that provides a way to represent abstract concepts—both deepens students' understanding of elementary topology and lays a solid foundation for future work in advanced topics.

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"This book is at the leading edge of what will likely become a major pedagogical trend in mathematics: teaching the fundamentals from a categorical perspective. "—David Spivak, Research Scientist at MIT, author of *Category Theory for the Sciences*

"As an algebraic topologist who has taught point-set topology from an implicitly category-theoretic viewpoint for many years, I was delighted to discover this beautifully written textbook."—Kathryn Hess, Professor, EPFL

"Bradley, Bryson, and Terilla make a compelling case for approaching category theory through point-set topology, imparting a lovely point of view that enlivens both subjects."—Emily Riehl, Associate Professor, Johns Hopkins University, author of *Categorical Homotopy Theory* and *Category Theory in Context*

"The categorical approach used is not only well motivated, but presented in a style that is very user-friendly." —Jim Stasheff, Professor Emeritus UNC-CH; Visiting Researcher at the University of Pennsylvania; coauthor of *Characteristic Classes*

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