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Profinite Semigroups and Symbolic Dynamics

This book describes the relation between profinite semigroups and symbolic dynamics. Profinite semigroups are topological semigroups which are compact and residually finite. In particular, free profinite semigroups can be seen as the completion of free semigroups with respect to the profinite metric. In this metric, two words are close if one needs a morphism on a large finite monoid to distinguish them. The main focus is on a natural correspondence between minimal shift spaces (closed shift-invariant sets of two-sided infinite words) and maximal J-classes (certain subsets of free profinite semigroups). This correspondence sheds light on many aspects of both profinite semigroups and symbolic dynamics. For example, the return words to a given word in a shift space can be related to the generators of the group of the corresponding J-class. The book is aimed at researchers and graduate students in mathematics or theoretical computer science.

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