

INTERDISCIPLINARY APPLIED MATHEMATICS

MATHEMATICAL BIOLOGY

There has been a long history of interaction between mathematics and physiology. This book looks in detail at a wide selection of mathematical models in physiology, showing how physiological problems can be formulated and studied mathematically, and how such models give rise to interesting and challenging mathematical questions. With its coverage of many recent models it gives an overview of the field, while many older models are also discussed, to put the modern work in context.

In this second edition the coverage of basic principles has been expanded to include such topics as stochastic differential equations, Markov models and Gibbs free energy, and the selection of models has also been expanded to include some of the basic models of fluid transport, respiration/perfusion, blood diseases, molecular motors, smooth muscle, neuroendocrine cells, the baroreceptor loop, turboglomerular oscillations, blood clotting and the retina.

Owing to this extensive coverage, the second edition is published in two volumes. This second volume deals with the physiology of systems and the first volume with the fundamental principles of cell physiology.

The book includes detailed illustrations and numerous exercises with selected solutions. The emphasis throughout is on the applications; because of this interdisciplinary approach, this book will be of interest to students and researchers, not only in mathematics, but also in bioengineering, physics, chemistry, biology, statistics and medicine.

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Reviews of the first edition:

"...probably the best book ever written on the interdisciplinary field of mathematical physiology."
—*Mathematical Reviews*, 2000

"In addition to being good reading, excellent pedagogy, and appealing science, the exposition is lucid and clear, and there are many good problem sets to choose from... Highly recommended."
—*Mathematical Biosciences*, 1999

"Both authors are seasoned experts in the field of mathematical physiology and particularly in the field of excitability, calcium dynamics and spiral waves. It directs students to become not merely skilled technicians in biological research but masters of the science."
—*SIAM*, 2004

The first edition was the winner of the 1998 Association of American Publishers "Best New Title in Mathematics."

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CONTENTS, II: Systems Physiology

Preface to the Second Edition	vii
Preface to the First Edition	ix
Acknowledgments	xiii
11 The Circulatory System	471
11.1 Blood Flow	473
11.2 Compliance	476
11.3 The Microcirculation and Filtration	479
11.4 Cardiac Output	482
11.5 Circulation	484
11.5.1 A Simple Circulatory System	484
11.5.2 A Linear Circulatory System	486
11.5.3 A Multicompartment Circulatory System	488
11.6 Cardiovascular Regulation	495
11.6.1 Autoregulation	497
11.6.2 The Baroreceptor Loop	500
11.7 Fetal Circulation	507
11.7.1 Pathophysiology of the Circulatory System	511

11.8	The Arterial Pulse	513
11.8.1	The Conservation Laws	513
11.8.2	The Windkessel Model	514
11.8.3	A Small-Amplitude Pressure Wave	516
11.8.4	Shock Waves in the Aorta	516
11.9	Exercises	521
12	The Heart	523
12.1	The Electrocardiogram	525
12.1.1	The Scalar ECG	525
12.1.2	The Vector ECG	526
12.2	Cardiac Cells	534
12.2.1	Purkinje Fibers	535
12.2.2	Sinoatrial Node	541
12.2.3	Ventricular Cells	543
12.2.4	Cardiac Excitation–Contraction Coupling	546
12.2.5	Common-Pool and Local-Control Models	548
12.2.6	The L-type Ca^{2+} Channel	550
12.2.7	The Ryanodine Receptor	551
12.2.8	The Na^{+} – Ca^{2+} Exchanger	552
12.3	Cellular Coupling	553
12.3.1	One-Dimensional Fibers	554
12.3.2	Propagation Failure	561
12.3.3	Myocardial Tissue: The Bidomain Model	566
12.3.4	Pacemakers	572
12.4	Cardiac Arrhythmias	583
12.4.1	Cellular Arrhythmias	584
12.4.2	Atrioventricular Node—Wenckebach Rhythms	586
12.4.3	Reentrant Arrhythmias	593
12.5	Defibrillation	604
12.5.1	The Direct Stimulus Threshold	608
12.5.2	The Defibrillation Threshold	610
12.6	Appendix: The Sawtooth Potential	613
12.7	Appendix: The Phase Equations	614
12.8	Appendix: The Cardiac Bidomain Equations	618
12.9	Exercises	622
13	Blood	627
13.1	Blood Plasma	628
13.2	Blood Cell Production	630
13.2.1	Periodic Hematological Diseases	632
13.2.2	A Simple Model of Blood Cell Growth	633
13.2.3	Peripheral or Local Control?	639

13.3	Erythrocytes	643
13.3.1	Myoglobin and Hemoglobin	643
13.3.2	Hemoglobin Saturation Shifts	648
13.3.3	Carbon Dioxide Transport	649
13.4	Leukocytes	652
13.4.1	Leukocyte Chemotaxis	653
13.4.2	The Inflammatory Response	655
13.5	Control of Lymphocyte Differentiation	665
13.6	Clotting	669
13.6.1	The Clotting Cascade	669
13.6.2	Clotting Models	671
13.6.3	<i>In Vitro</i> Clotting and the Spread of Inhibition	671
13.6.4	Platelets	675
13.7	Exercises	678
14	Respiration	683
14.1	Capillary–Alveoli Gas Exchange	684
14.1.1	Diffusion Across an Interface	684
14.1.2	Capillary–Alveolar Transport	685
14.1.3	Carbon Dioxide Removal	688
14.1.4	Oxygen Uptake	689
14.1.5	Carbon Monoxide Poisoning	692
14.2	Ventilation and Perfusion	694
14.2.1	The Oxygen–Carbon Dioxide Diagram	698
14.2.2	Respiratory Exchange Ratio	698
14.3	Regulation of Ventilation	701
14.3.1	A More Detailed Model of Respiratory Regulation	706
14.4	The Respiratory Center	708
14.4.1	A Simple Mutual Inhibition Model	710
14.5	Exercises	714
15	Muscle	717
15.1	Crossbridge Theory	719
15.2	The Force–Velocity Relationship: The Hill Model	724
15.2.1	Fitting Data	726
15.2.2	Some Solutions of the Hill Model	727
15.3	A Simple Crossbridge Model: The Huxley Model	730
15.3.1	Isotonic Responses	737
15.3.2	Other Choices for Rate Functions	738
15.4	Determination of the Rate Functions	739
15.4.1	A Continuous Binding Site Model	739
15.4.2	A General Binding Site Model	741
15.4.3	The Inverse Problem	742

15.5	The Discrete Distribution of Binding Sites	747
15.6	High Time-Resolution Data	748
15.6.1	High Time-Resolution Experiments	748
15.6.2	The Model Equations	749
15.7	In Vitro Assays	755
15.8	Smooth Muscle	756
15.8.1	The Hai-Murphy Model	756
15.9	Large-Scale Muscle Models	759
15.10	Molecular Motors	759
15.10.1	Brownian Ratchets	760
15.10.2	The Tilted Potential	765
15.10.3	Flashing Ratchets	767
15.11	Exercises	770
16	The Endocrine System	773
16.1	The Hypothalamus and Pituitary Gland	775
16.1.1	Pulsatile Secretion of Luteinizing Hormone	777
16.1.2	Neural Pulse Generator Models	779
16.2	Ovulation in Mammals	784
16.2.1	A Model of the Menstrual Cycle	784
16.2.2	The Control of Ovulation Number	788
16.2.3	Other Models of Ovulation	802
16.3	Insulin and Glucose	803
16.3.1	Insulin Sensitivity	804
16.3.2	Pulsatile Insulin Secretion	806
16.4	Adaptation of Hormone Receptors	813
16.5	Exercises	816
17	Renal Physiology	821
17.1	The Glomerulus	821
17.1.1	Autoregulation and Tubuloglomerular Oscillations	825
17.2	Urinary Concentration: The Loop of Henle	831
17.2.1	The Countercurrent Mechanism	836
17.2.2	The Countercurrent Mechanism in Nephrons	837
17.3	Models of Tubular Transport	848
17.4	Exercises	849
18	The Gastrointestinal System	851
18.1	Fluid Absorption	851
18.1.1	A Simple Model of Fluid Absorption	853
18.1.2	Standing-Gradient Osmotic Flow	857
18.1.3	Uphill Water Transport	864

18.2	Gastric Protection	866
18.2.1	A Steady-State Model	867
18.2.2	Gastric Acid Secretion and Neutralization	873
18.3	Coupled Oscillators in the Small Intestine	874
18.3.1	Temporal Control of Contractions	874
18.3.2	Waves of Electrical Activity	875
18.3.3	Models of Coupled Oscillators	878
18.3.4	Interstitial Cells of Cajal	887
18.3.5	Biophysical and Anatomical Models	888
18.4	Exercises	890
19	The Retina and Vision	893
19.1	Retinal Light Adaptation	895
19.1.1	Weber's Law and Contrast Detection	897
19.1.2	Intensity-Response Curves and the Naka-Rushton Equation	898
19.2	Photoreceptor Physiology	902
19.2.1	The Initial Cascade	905
19.2.2	Light Adaptation in Cones	907
19.3	A Model of Adaptation in Amphibian Rods	912
19.3.1	Single-Photon Responses	915
19.4	Lateral Inhibition	917
19.4.1	A Simple Model of Lateral Inhibition	919
19.4.2	Photoreceptor and Horizontal Cell Interactions	921
19.5	Detection of Motion and Directional Selectivity	926
19.6	Receptive Fields	929
19.7	The Pupil Light Reflex	933
19.7.1	Linear Stability Analysis	935
19.8	Appendix: Linear Systems Theory	936
19.9	Exercises	939
20	The Inner Ear	943
20.1	Frequency Tuning	946
20.1.1	Cochlear Macromechanics	947
20.2	Models of the Cochlea	949
20.2.1	Equations of Motion for an Incompressible Fluid	949
20.2.2	The Basilar Membrane as a Harmonic Oscillator	950
20.2.3	An Analytical Solution	952
20.2.4	Long-Wave and Short-Wave Models	953
20.2.5	More Complex Models	962
20.3	Electrical Resonance in Hair Cells	962
20.3.1	An Electrical Circuit Analogue	964
20.3.2	A Mechanistic Model of Frequency Tuning	966

20.4	The Nonlinear Cochlear Amplifier	969
20.4.1	Negative Stiffness, Adaptation, and Oscillations	969
20.4.2	Nonlinear Compression and Hopf Bifurcations	971
20.5	Exercises	973
Appendix: Units and Physical Constants		A-1
References		R-1
Index		I-1

CONTENTS, I: Cellular Physiology

Preface to the Second Edition		vii
Preface to the First Edition		ix
Acknowledgments		xiii
1	Biochemical Reactions	1
1.1	The Law of Mass Action	1
1.2	Thermodynamics and Rate Constants	3
1.3	Detailed Balance	6
1.4	Enzyme Kinetics	7
1.4.1	The Equilibrium Approximation	8
1.4.2	The Quasi-Steady-State Approximation	9
1.4.3	Enzyme Inhibition	12
1.4.4	Cooperativity	15
1.4.5	Reversible Enzyme Reactions	20
1.4.6	The Goldbeter-Koshland Function	21
1.5	Glycolysis and Glycolytic Oscillations	23
1.6	Appendix: Math Background	33
1.6.1	Basic Techniques	35
1.6.2	Asymptotic Analysis	37
1.6.3	Enzyme Kinetics and Singular Perturbation Theory	39
1.7	Exercises	42
2	Cellular Homeostasis	49
2.1	The Cell Membrane	49
2.2	Diffusion	51
2.2.1	Fick's Law	52
2.2.2	Diffusion Coefficients	53
2.2.3	Diffusion Through a Membrane: Ohm's Law	54

2.2.4	Diffusion into a Capillary	55
2.2.5	Buffered Diffusion	55
2.3	Facilitated Diffusion	58
2.3.1	Facilitated Diffusion in Muscle Respiration	61
2.4	Carrier-Mediated Transport	63
2.4.1	Glucose Transport	64
2.4.2	Symports and Antiports	67
2.4.3	Sodium–Calcium Exchange	69
2.5	Active Transport	73
2.5.1	A Simple ATPase	74
2.5.2	Active Transport of Charged Ions	76
2.5.3	A Model of the $\text{Na}^+ - \text{K}^+$ ATPase	77
2.5.4	Nuclear Transport	79
2.6	The Membrane Potential	80
2.6.1	The Nernst Equilibrium Potential	80
2.6.2	Gibbs–Donnan Equilibrium	82
2.6.3	Electrodifusion: The Goldman–Hodgkin–Katz Equations	83
2.6.4	Electrical Circuit Model of the Cell Membrane	86
2.7	Osmosis	88
2.8	Control of Cell Volume	90
2.8.1	A Pump–Leak Model	91
2.8.2	Volume Regulation and Ionic Transport	98
2.9	Appendix: Stochastic Processes	103
2.9.1	Markov Processes	103
2.9.2	Discrete-State Markov Processes	105
2.9.3	Numerical Simulation of Discrete Markov Processes	107
2.9.4	Diffusion	109
2.9.5	Sample Paths; the Langevin Equation	110
2.9.6	The Fokker–Planck Equation and the Mean First Exit Time	111
2.9.7	Diffusion and Fick’s Law	114
2.10	Exercises	115
3	Membrane Ion Channels	121
3.1	Current–Voltage Relations	121
3.1.1	Steady-State and Instantaneous Current–Voltage Relations	123
3.2	Independence, Saturation, and the Ussing Flux Ratio	125
3.3	Electrodifusion Models	128
3.3.1	Multi-Ion Flux: The Poisson–Nernst–Planck Equations	129
3.4	Barrier Models	134
3.4.1	Nonsaturating Barrier Models	136
3.4.2	Saturating Barrier Models: One-Ion Pores	139
3.4.3	Saturating Barrier Models: Multi-Ion Pores	143
3.4.4	Electrogenic Pumps and Exchangers	145

3.5	Channel Gating	147
3.5.1	A Two-State K^+ Channel	148
3.5.2	Multiple Subunits	149
3.5.3	The Sodium Channel	150
3.5.4	Agonist-Controlled Ion Channels	152
3.5.5	Drugs and Toxins	153
3.6	Single-Channel Analysis	155
3.6.1	Single-Channel Analysis of a Sodium Channel	155
3.6.2	Single-Channel Analysis of an Agonist-Controlled Ion Channel	158
3.6.3	Comparing to Experimental Data	160
3.7	Appendix: Reaction Rates	162
3.7.1	The Boltzmann Distribution	163
3.7.2	A Fokker-Planck Equation Approach	165
3.7.3	Reaction Rates and Kramers' Result	166
3.8	Exercises	170
4	Passive Electrical Flow in Neurons	175
4.1	The Cable Equation	177
4.2	Dendritic Conduction	180
4.2.1	Boundary Conditions	181
4.2.2	Input Resistance	182
4.2.3	Branching Structures	182
4.2.4	A Dendrite with Synaptic Input	185
4.3	The Rall Model of a Neuron	187
4.3.1	A Semi-Infinite Neuron with a Soma	187
4.3.2	A Finite Neuron and Soma	189
4.3.3	Other Compartmental Models	192
4.4	Appendix: Transform Methods	192
4.5	Exercises	193
5	Excitability	195
5.1	The Hodgkin-Huxley Model	196
5.1.1	History of the Hodgkin-Huxley Equations	198
5.1.2	Voltage and Time Dependence of Conductances	200
5.1.3	Qualitative Analysis	210
5.2	The FitzHugh-Nagumo Equations	216
5.2.1	The Generalized FitzHugh-Nagumo Equations	219
5.2.2	Phase-Plane Behavior	220
5.3	Exercises	223

6	Wave Propagation in Excitable Systems	229
6.1	Brief Overview of Wave Propagation	229
6.2	Traveling Fronts	231
6.2.1	The Bistable Equation	231
6.2.2	Myelination	236
6.2.3	The Discrete Bistable Equation	238
6.3	Traveling Pulses	242
6.3.1	The FitzHugh–Nagumo Equations	242
6.3.2	The Hodgkin–Huxley Equations	250
6.4	Periodic Wave Trains	252
6.4.1	Piecewise-Linear FitzHugh–Nagumo Equations	253
6.4.2	Singular Perturbation Theory	254
6.4.3	Kinematics	256
6.5	Wave Propagation in Higher Dimensions	257
6.5.1	Propagating Fronts	258
6.5.2	Spatial Patterns and Spiral Waves	262
6.6	Exercises	268
7	Calcium Dynamics	273
7.1	Calcium Oscillations and Waves	276
7.2	Well-Mixed Cell Models: Calcium Oscillations	281
7.2.1	Influx	282
7.2.2	Mitochondria	282
7.2.3	Calcium Buffers	282
7.2.4	Calcium Pumps and Exchangers	283
7.2.5	IP ₃ Receptors	285
7.2.6	Simple Models of Calcium Dynamics	293
7.2.7	Open- and Closed-Cell Models	296
7.2.8	IP ₃ Dynamics	298
7.2.9	Ryanodine Receptors	301
7.3	Calcium Waves	303
7.3.1	Simulation of Spiral Waves in <i>Xenopus</i>	306
7.3.2	Traveling Wave Equations and Bifurcation Analysis	307
7.4	Calcium Buffering	309
7.4.1	Fast Buffers or Excess Buffers	310
7.4.2	The Existence of Buffered Waves	313
7.5	Discrete Calcium Sources	315
7.5.1	The Fire–Diffuse–Fire Model	318
7.6	Calcium Puffs and Stochastic Modeling	321
7.6.1	Stochastic IPR Models	323
7.6.2	Stochastic Models of Calcium Waves	324

7.7	Intercellular Calcium Waves	32
7.7.1	Mechanically Stimulated Intercellular Ca^{2+} Waves	32
7.7.2	Partial Regeneration	33
7.7.3	Coordinated Oscillations in Hepatocytes	33
7.8	Appendix: Mean Field Equations	33
7.8.1	Microdomains	33
7.8.2	Homogenization; Effective Diffusion Coefficients	33
7.8.3	Bidomain Equations	34
7.9	Exercises	34
8	Intercellular Communication	347
8.1	Chemical Synapses	348
8.1.1	Quantal Nature of Synaptic Transmission	349
8.1.2	Presynaptic Voltage-Gated Calcium Channels	352
8.1.3	Presynaptic Calcium Dynamics and Facilitation	358
8.1.4	Neurotransmitter Kinetics	364
8.1.5	The Postsynaptic Membrane Potential	370
8.1.6	Agonist-Controlled Ion Channels	371
8.1.7	Drugs and Toxins	373
8.2	Gap Junctions	373
8.2.1	Effective Diffusion Coefficients	374
8.2.2	Homogenization	376
8.2.3	Measurement of Permeabilities	377
8.2.4	The Role of Gap-Junction Distribution	377
8.3	Exercises	383
9	Neuroendocrine Cells	385
9.1	Pancreatic Cells	386
9.1.1	Bursting in the Pancreatic Cell	386
9.1.2	ER Calcium as a Slow Controlling Variable	392
9.1.3	Slow Bursting and Glycolysis	399
9.1.4	Bursting in Clusters	403
9.1.5	A Qualitative Bursting Model	410
9.1.6	Bursting Oscillations in Other Cell Types	412
9.2	Hypothalamic and Pituitary Cells	419
9.2.1	The Gonadotroph	419
9.3	Exercises	424
10	Regulation of Cell Function	427
10.1	Regulation of Gene Expression	428
10.1.1	The <i>trp</i> Repressor	429
10.1.2	The <i>lac</i> Operon	432

10.2	Circadian Clocks	438
10.3	The Cell Cycle	442
10.3.1	A Simple Generic Model	445
10.3.2	Fission Yeast	452
10.3.3	A Limit Cycle Oscillator in the <i>Xenopus</i> Oocyte	461
10.3.4	Conclusion	468
10.4	Exercises	468

Appendix: Units and Physical Constants	A-1
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References	R-1
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Index	I-1
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