Advances in Computer Vision and Pattern Recognition
Sing Bing Kang *Series Editor*

Guy Gilboa

Nonlinear Eigenproblems in Image Processing and Computer Vision

This unique text/reference presents a fresh look at nonlinear processing through nonlinear eigenvalue analysis, highlighting how one-homogeneous convex functionals can induce nonlinear operators that can be analyzed within an eigenvalue framework. The text opens with an introduction to the mathematical background, together with a summary of classical variational algorithms for vision. This is followed by a focus on the foundations and applications of the new multi-scale representation based on nonlinear eigenproblems. The book then concludes with a discussion of new numerical techniques for finding nonlinear eigenfunctions, and promising research directions beyond the convex case.

Topics and features:

- Introduces the classical Fourier transform and its associated operator and energy, and asks how these concepts can be generalized in the nonlinear case
- Reviews the basic mathematical notion, briefly outlining the use of variational and flow-based methods to solve image-processing and computer vision algorithms
- Describes the properties of the total variation (TV) functional, and how the concept of nonlinear eigenfunctions relate to convex functionals
- Provides a spectral framework for one-homogeneous functionals, and applies this framework for denoising, texture processing and image fusion
- Proposes novel ways to solve the nonlinear eigenvalue problem using special flows that converge to eigenfunctions
- Examines graph-based and nonlocal methods, for which a TV eigenvalue analysis gives rise to strong segmentation, clustering and classification algorithms
- Presents an approach to generalizing the nonlinear spectral concept beyond the convex case, based on pixel decay analysis
- Discusses relations to other branches of image processing, such as wavelets and dictionary based methods

This original work offers fascinating new insights into established signal processing techniques, integrating deep mathematical concepts from a range of different fields, which will be of great interest to all researchers involved with image processing and computer vision applications, as well as computations for more general scientific problems.

Dr. Guy Gilboa is an Assistant Professor in the Electrical Engineering Department at Technion – Israel Institute of Technology, Haifa, Israel.

Computer Science

ISBN 978-3-319-75846-6

9 783319 758466

▸ **springer.com**

1 **Mathematical Preliminaries** 1
1.1 Reminder of Very Basic Operators and Definitions 1
1.1.1 Integration by Parts (Reminder) 2
1.1.2 Distributions (Reminder) 3
1.2 Some Standard Spaces 3
1.3 Euler–Lagrange 4
1.3.1 E–L of Some Functionals 5
1.3.2 Some Useful Examples 6
1.3.3 E–L of Common Fidelity Terms 7
1.3.4 Norms Without Derivatives 7
1.3.5 Seminorms with Derivatives 8
1.4 Convex Functionals 9
1.4.1 Convex Function and Functional 9
1.4.2 Why Convex Functions Are Good? 10
1.4.3 Subdifferential 10
1.4.4 Duality—Legendre–Fenchel Transform 11
1.5 One-Homogeneous Functionals 11
1.5.1 Definition and Basic Properties 11
References 14

2 **Variational Methods in Image Processing** 15
2.1 Variation Modeling by Regularizing Functionals 15
2.1.1 Regularization Energies and Their Respective E-L 17
2.2 Nonlinear PDEs 18
2.2.1 Gaussian Scale Space 18
2.2.2 Perona–Malik Nonlinear Diffusion 19
2.2.3 Weickert's Anisotropic Diffusion 20
2.2.4 Steady-State Solution 21
2.2.5 Inverse Scale Space 22

2.3 Optical Flow and Registration 22
2.3.1 Background 22
2.3.2 Early Attempts for Solving the Optical Flow Problem 24
2.3.3 Modern Optical Flow Techniques 25
2.4 Segmentation and Clustering 26
2.4.1 The Goal of Segmentation 26
2.4.2 Mumford–Shah 26
2.4.3 Chan–Vese Model 27
2.4.4 Active Contours 28
2.5 Patch-Based and Nonlocal Models 29
2.5.1 Background 29
2.5.2 Graph Laplacian 30
2.5.3 A Nonlocal Mathematical Framework 31
2.5.4 Basic Models 34
References 35

3 Total Variation and Its Properties 39
3.1 Strong and Weak Definitions 39
3.2 Co-area Formula 40
3.3 Definition of BV 40
3.4 Basic Concepts Related to TV 41
3.4.1 Isotropic and Anisotropic TV 41
3.4.2 ROF, TV-L1, and TV Flow 41
References 43

4 Eigenfunctions of One-Homogeneous Functionals 45
4.1 Introduction 45
4.2 One-Homogeneous Functionals 46
4.3 Properties of Eigenfunction 47
4.4 Eigenfunctions of TV 48
4.4.1 Explicit TV Eigenfunctions in 1D 49
4.5 Pseudo-Eigenfunctions 53
4.5.1 Measure of Affinity of Nonlinear Eigenfunctions 53
References 56

5 Spectral One-Homogeneous Framework 59
5.1 Preliminary Definitions and Settings 59
5.2 Spectral Representations 59
5.2.1 Scale Space Representation 59
5.3 Signal Processing Analogy 61
5.3.1 Nonlinear Ideal Filters 63
5.3.2 Spectral Response 63

5.4 Theoretical Analysis and Properties . . . 66
5.4.1 Variational Representation . . . 66
5.4.2 Scale Space Representation . . . 70
5.4.3 Inverse Scale Space Representation . . . 71
5.4.4 Definitions of the Power Spectrum . . . 73
5.5 Analysis of the Spectral Decompositions . . . 74
5.5.1 Basic Conditions on the Regularization . . . 74
5.5.2 Connection Between Spectral Decompositions . . . 77
5.5.3 Orthogonality of the Spectral Components . . . 82
5.5.4 Nonlinear Eigendecompositions . . . 88
References . . . 90

6 Applications Using Nonlinear Spectral Processing . . . 93
6.1 Generalized Filters . . . 93
6.1.1 Basic Image Manipulation . . . 93
6.2 Simplification and Denoising . . . 93
6.2.1 Denoising with Trained Filters . . . 98
6.3 Multiscale and Spatially Varying Filtering Horesh–Gilboa . . . 100
6.4 Face Fusion and Style Transfer . . . 102
References . . . 104

7 Numerical Methods for Finding Eigenfunctions . . . 107
7.1 Linear Methods . . . 107
7.2 Hein–Buhler . . . 108
7.3 Nossek–Gilboa . . . 109
7.3.1 Flow Main Properties . . . 110
7.3.2 Inverse Flow . . . 113
7.3.3 Discrete Time Flow . . . 114
7.3.4 Properties of the Discrete Flow . . . 114
7.3.5 Normalized Flow . . . 117
7.4 Aujol et al. Method . . . 120
References . . . 122

8 Graph and Nonlocal Framework . . . 123
8.1 Graph Total Variation Analysis . . . 123
8.2 Graph P-Laplacian Operators . . . 124
8.3 The Cheeger Cut . . . 126
8.4 The Graph 1-Laplacian . . . 127
8.5 The *p*-flow . . . 129
8.5.1 Flow Main Properties . . . 129
8.5.2 Numerical Scheme . . . 130
8.5.3 Algorithm . . . 131
References . . . 132

9 Beyond Convex Analysis—Decompositions with Nonlinear Flows 133
9.1 General Decomposition Based on Nonlinear Denoisers 133
9.1.1 A Spectral Transform 134
9.1.2 Inverse Transform, Spectrum, and Filtering 134
9.1.3 Determining the Decay Profiles 135
9.2 Blind Spectral Decomposition 136
9.3 Theoretical Analysis 138
9.3.1 Generalized Eigenvectors 139
9.3.2 Relation to Known Transforms 139
References 140

10 Relations to Other Decomposition Methods 141
10.1 Decomposition into Eigenfunctions 141
10.2 Wavelets and Hard Thresholding 142
10.2.1 Haar Wavelets 142
10.3 Rayleigh Quotients and SVD Decomposition 143
10.4 Sparse Representation by Eigenfunctions 148
10.4.1 Total Variation Dictionaries 149
10.4.2 Dictionaries from One-Homogeneous Functionals 149
References 150

11 Future Directions 151
11.1 Spectral Total Variation Local Time Signatures for Image Manipulation and Fusion 151
11.2 Spectral AATV (Adapted Anisotropic Total Variation) and the AATV Eigenfunction Analysis 153
11.3 TV Spectral Hashing 155
11.4 Some Open Problems 157
Reference 158

Appendix A: Numerical Schemes 159

Glossary 169

Index 171