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Nontechnical chapters are indicated with a star *; Discussion chapters are indicated with a †; adaptation from published ("peer-reviewed") papers with a ‡.

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