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As a matter of fact, the reader is introduced to the historical work of quantum mechanics relating between quantum theory, probability, and matrix theory, including the basic theory of von Neumann algebras and the mathematical foundations of quantum theory. This book would serve as a good reference for an advanced graduate student who wish to acquire the tools that underlie the physical aspects of quantum mechanics and as the able physicist eager to solidify their understanding of the mathematical underpinning of quantum theories. Several examples and solved exercises accompany the algebraic approach—most of which carefully demonstrated—and physical motivations are provided for every mathematical notion. The said I must point out that this is not a manual on (higher) quantum mechanics. There are many (quite good) books that deal instead in advanced technical aspects like Schrödinger equation using the proper machinery of Hilbert, for which reason those exercises are found here.

Some of the proofs omitted appear in [Mor15], other parts are completely new. For instance, Chap. 2, the last section of Chap. 4 and some material in Chap. 8. Despite a good degree of ideological variation, [Mor15] is more complete theoretically, but its 500+ pages do not make it suitable for a single semester course. Most of the proofs here are in fact novel, because they were developed independently to reflect the relative scarceness of the technical details that are in a comment.

The book is organized as follows: