This book provides researchers and graduate students with a thorough introduction to the variational analysis of nonlinear problems described by nonlocal operators. The authors give a systematic treatment of the basic mathematical theory and constructive methods for these classes of nonlinear equations, plus their application to various processes arising in the applied sciences. The equations are examined from several viewpoints, with the calculus of variations as the unifying theme. Part I begins the book with some basic facts about fractional Sobolev spaces. Part II is dedicated to the analysis of fractional elliptic problems involving subcritical nonlinearities, via classical variational methods and other novel approaches. Finally, Part III contains a selection of recent results on critical fractional equations.

A careful balance is struck between rigorous mathematics and physical applications, allowing readers to see how these diverse topics relate to other important areas, including topology, functional analysis, mathematical physics, and potential theory.

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