

This textbook is specially addressed to the needs of the large community of applied mathematicians, physicists, engineers etc., who are interested to study the problems of mathematical physics in general and their approximate solutions on computer in particular. The book grew out of such a course in the Department of Mathematics, Indian Institute of Technology, Delhi. Almost all basic results of theory of distributions are contained in the book. Hence it can be read as an introduction to advanced treatises on distributions.

The book contains almost all topics of Sobolev spaces which are essential for the study of elliptic boundary value problems and their finite element analysis. Moreover the nice properties of Sobolev spaces on $\mathbb{R}^n$, their extensions to those on $\Omega \neq \mathbb{R}^n$, imbedding results with compactness properties, Sobolev spaces on manifolds Γ (boundary of Ω) and trace theorems for $\mathbb{R}^n, \mathbb{R}^n_+, C^m$ -regular domains Ω in $\mathbb{R}^n$ and polygonal domains Ω in $\mathbb{R}^2$ are also treated systematically.

Additional topics, such as methods of construction of elementary solutions of differential operators, distributional derivatives of discontinuous piecewise smooth functions of several variables, delta-convergent sequences of smooth functions, Fourier series of periodic distributions, convolution system of equations and Laplace transforms etc., have been included along with many interesting examples. Physical interpretation of the duality principle and discussion on physical distributions versus mathematical distributions and application of convolution of distributions for the analysis of the RLC circuit of electrical engineering can be found in the book.

- ▶ Innumerable examples are given with all intermediate steps and explanations, which are usually not given in advanced treatises.
- ▶ Proofs include intermediate steps and necessary explanations to make them easily understandable.
- ▶ The book is written by an applied person for the applied community and will serve as a reference-cum-text book of the applied people.



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