

Classical Fourier Analysis

Third Edition

The main goal of this text is to present the theoretical foundation of the field of Fourier analysis on Euclidean spaces. It covers *classical* topics such as interpolation, Fourier series, the Fourier transform, maximal functions, singular integrals, and Littlewood–Paley theory. The primary readership is intended to be graduate students in mathematics who have completed courses in real and complex analysis. The coverage of topics and exposition style are designed to leave no gaps in understanding and stimulate further study.

This third edition includes several new sections as well as a new chapter on Weighted Inequalities, which has been moved from GTM 250, 2nd Edition. Countless corrections and improvements have been made to the material from the second edition. Additions and improvements include: more examples and applications, new and more relevant hints for the existing exercises, new exercises, and improved references.

Reviews from the Second Edition

The books cover a large amount of mathematics. They are certainly a valuable and useful addition to the existing literature and can serve as textbooks or as reference books. Students will especially appreciate the extensive collection of exercises. **Andreas Seager,**
Mathematical Reviews

This book is very interesting and useful. It is not only a good textbook, but also an indispensable and valuable reference for researchers who are working on analysis and partial differential equations. The readers will certainly benefit a lot from the detailed proofs and the numerous exercises. **Yang Dachun, zbMATH**

Mathematics

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