

Prime numbers have fascinated mathematicians since the time of Euclid. This book presents some of our best tools to capture the properties of these fundamental objects, beginning with the most basic notions of asymptotic estimates and arriving at the forefront of mathematical research. Detailed proofs of the recent spectacular advances on small and large gaps between primes are made accessible for the first time in textbook form. Some other highlights include an introduction to probabilistic methods, a detailed study of sieves, and elements of the theory of pretentious multiplicative functions leading to a proof of Linnik's theorem.



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Throughout, the emphasis has been placed on explaining the main ideas rather than the most general results available. As a result, several methods are presented in terms of concrete examples that simplify technical details, and theorems are stated in a form that facilitates the understanding of their proof at the cost of sacrificing some generality. Each chapter concludes with numerous exercises of various levels of difficulty aimed to exemplify the material, as well as to expose the readers to more advanced topics and point them to further reading sources.

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Preface	vii
Notation	xi
And then there were infinitely many	1
Part 1. First principles	
Chapter 1. Asymptotic estimates	8
Chapter 2. Combinatorial ways to count primes	27
Chapter 3. The Dirichlet convolution	35
Chapter 4. Dirichlet series	44
Part 2. Methods of complex and harmonic analysis	
Chapter 5. An explicit formula for counting primes	52
Chapter 6. The Riemann zeta function	62
Chapter 7. The Perron inversion formula	70
Chapter 8. The Prime Number Theorem	84
Chapter 9. Dirichlet characters	95
Chapter 10. Fourier analysis on finite abelian groups	100
Chapter 11. Dirichlet L -functions	110
Chapter 12. The Prime Number Theorem for arithmetic progressions	118

Part 3. Multiplicative functions and the anatomy of integers

Chapter 13.	Primes and multiplicative functions	130
Chapter 14.	Evolution of sums of multiplicative functions	143
Chapter 15.	The distribution of multiplicative functions	157
Chapter 16.	Large deviations	164

Part 4. Sieve methods

Chapter 17.	Twin primes	174
Chapter 18.	The axioms of sieve theory	182
Chapter 19.	The Fundamental Lemma of Sieve Theory	192
Chapter 20.	Applications of sieve methods	206
Chapter 21.	Selberg's sieve	213
Chapter 22.	Sieving for zero-free regions	222

Part 5. Bilinear methods

Chapter 23.	Vinogradov's method	234
Chapter 24.	Ternary arithmetic progressions	250
Chapter 25.	Bilinear forms and the large sieve	259
Chapter 26.	The Bombieri-Vinogradov theorem	277
Chapter 27.	The least prime in an arithmetic progression	287

Part 6. Local aspects of the distribution of primes

Chapter 28.	Small gaps between primes	300
Chapter 29.	Large gaps between primes	317
Chapter 30.	Irregularities in the distribution of primes	329

Appendices

Appendix A.	The Riemann-Stieltjes integral	336
Appendix B.	The Fourier and the Mellin transforms	338
Appendix C.	The method of moments	341
Bibliography		344
Index		354