

STABLE SOLUTIONS OF ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

Stable solutions are ubiquitous in differential equations. They represent meaningful solutions from a physical point of view and appear in many applications, including mathematical physics (combustion, phase transition theory) and geometry (minimal surfaces).

Stable Solutions of Elliptic Partial Differential Equations offers a self-contained presentation of the notion of stability in elliptic partial differential equations (PDEs). The central questions of regularity and classification of stable solutions are treated at length. Specialists will find a summary of the most recent developments of the theory, such as nonlocal and higher-order equations. For beginners, the book walks you through the fine versions of the maximum principle, the standard regularity theory for linear elliptic equations, and the fundamental functional inequalities commonly used in this field. The text also includes two additional topics: the inverse-square potential and some background material on submanifolds of Euclidean space.

Features

- Reviews the fundamentals of elliptic PDE theory, including geometric aspects
- Provides a thorough look at stable solutions, from basic properties to the most recent developments
- Explores advanced topics, such as singular solutions and level set analysis
- Contains an introduction to a celebrated conjecture of Ennio de Giorgi
- Includes both standard and challenging exercises



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