

This book is an introduction to the theory of partial differential operators. It assumes that the reader has a knowledge of introductory functional analysis, up to the spectral theorem for bounded linear operators on Banach spaces. However, it describes the theory of Fourier transforms and distributions as far as is needed to analyse the spectrum of any constant coefficient partial differential operator. A completely new proof of the spectral theorem for unbounded self-adjoint operators is followed by its application to a variety of second order elliptic differential operators, from those with discrete spectra to Schrödinger operators acting on $L^2(\mathbf{R}^N)$. The book contains a detailed account of the application of variational methods to estimate the eigenvalues of operators with measurable coefficients defined by the use of quadratic form techniques.

This book could be used either for self-study or as a course text, and aims to lead the reader to the more advanced literature on the subject.

Cambridge Studies in Advanced Mathematics

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