

A concise yet elementary introduction to measure and integration theory, which are vital in many areas of mathematics, including analysis, probability, mathematical physics and finance. In this highly successful textbook the core ideas of measure and integration are explored, and martingales are used to develop the theory further. Additional topics are also covered such as: Jacobi's transformation theorem; the Radon–Nikodym theorem; differentiation of measures and Hardy–Littlewood maximal functions.

In this second edition, readers will find newly added chapters on Hausdorff measures, Fourier analysis, vague convergence, and classical proofs of the Radon–Nikodym and Riesz representation theorems. All proofs are carefully worked out with utmost clarity to ensure full understanding of the material and its background.

Requiring few prerequisites, this book is a suitable text for undergraduate lecture courses or self-study. Numerous illustrations and over 400 exercises help to consolidate and broaden the reader's knowledge. Full solutions to all exercises are available on the author's webpage at www.motapa.de.

From the reviews of the First Edition:

'Overall, this well-written and carefully structured book is an excellent choice for an undergraduate course on measure and integration theory.'

Journal of the American Statistical Association

'I have not seen some of the topics that are mentioned above ... treated successfully at undergraduate level before, and the book is worth having for these alone ... [it] has the potential to revitalize the way that measure theory is taught.'

Journal of the Royal Statistical Society

'I believe this to be a great book for self-study as well as for course use. The book is ideal for future probabilists as well as statisticians, and can serve as a good introduction for mathematicians interested in measure theory.'

MAA Reviews

'This book will remain a good reference on the subject for years to come.'

Mathematical Reviews

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