

This book develops the theory of global attractors for a class of parabolic PDEs that includes reaction–diffusion equations and the Navier–Stokes equations, two examples that are treated in detail. A lengthy chapter on Sobolev spaces provides the framework that allows a rigorous treatment of existence and uniqueness of solutions for both linear time-independent problems (Poisson’s equation) and the nonlinear evolution equations that generate the infinite-dimensional dynamical systems of the title. Attention then turns to the global attractor, a finite-dimensional subset of the infinite-dimensional phase space that determines the asymptotic dynamics. In particular, the concluding chapters investigate in what sense the dynamics restricted to the attractor are themselves “finite-dimensional.”

The book is intended as a didactic text for first-year graduate students and assumes only a basic knowledge of elementary functional analysis.

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Cover illustration: A two-dimensional inertial manifold for the three-dimensional Galerkin truncation of the Chaffee–Infante reaction-diffusion equation, found using a computational version of the graph transform method.

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