

This book is a graduate text on the incompressible Navier-Stokes system, which is of fundamental importance in mathematical fluid mechanics as well as in engineering applications. The goal is to give a rapid exposition on the existence, uniqueness, and regularity of its solutions, with a focus on the regularity problem. To fit into a one-year course for students who have already mastered the basics of PDE theory, many auxiliary results have been described with references but without proofs, and several topics were omitted. Most chapters end with a selection of problems for the reader.

After an introduction and a careful study of weak, strong, and mild solutions, the reader is introduced to partial regularity. The coverage of boundary value problems, self-similar solutions, the uniform L^3 class including the celebrated Escauriaza-Seregin-Šverák Theorem, and axisymmetric flows in later chapters are unique features of this book that are less explored in other texts.

The book can serve as a textbook for a course, as a self-study source for people who already know some PDE theory and wish to learn more about Navier-Stokes equations, or as a reference for some of the important recent developments in the area.

The book is an excellent contribution to the literature concerning the mathematical analysis of the incompressible Navier-Stokes equations. It provides a very good introduction to the subject, covering several important directions, and also presents a number of recent results, with an emphasis on non-perturbative regimes. The book is well written, and both beginners and experts will benefit from it. It can also provide great material for a graduate course.

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