

# Contents

|          |                                                                                      |           |
|----------|--------------------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction .....</b>                                                            | <b>1</b>  |
| 1.1      | What Is a Time Series?.....                                                          | 1         |
| 1.2      | Time Series Versus iid Data.....                                                     | 2         |
| <b>2</b> | <b>Typical Assumptions .....</b>                                                     | <b>5</b>  |
| 2.1      | Fundamental Properties .....                                                         | 5         |
| 2.1.1    | Ergodic Property with a Constant Limit .....                                         | 5         |
| 2.1.2    | Strict Stationarity .....                                                            | 7         |
| 2.1.3    | Weak Stationarity .....                                                              | 8         |
| 2.1.4    | Weak Stationarity and Hilbert Spaces.....                                            | 11        |
| 2.1.5    | Ergodic Processes .....                                                              | 32        |
| 2.1.6    | Sufficient Conditions for the a.s. Ergodic Property<br>with a Constant Limit .....   | 34        |
| 2.1.7    | Sufficient Conditions for the $L^2$ -Ergodic Property<br>with a Constant Limit ..... | 35        |
| 2.2      | Specific Assumptions .....                                                           | 39        |
| 2.2.1    | Gaussian Processes .....                                                             | 39        |
| 2.2.2    | Linear Processes in $L^2(\Omega)$ .....                                              | 40        |
| 2.2.3    | Linear Processes with $E(X_t^2) = \infty$ .....                                      | 44        |
| 2.2.4    | Multivariate Linear Processes .....                                                  | 48        |
| 2.2.5    | Invertibility .....                                                                  | 49        |
| 2.2.6    | Restrictions on the Dependence Structure .....                                       | 63        |
| <b>3</b> | <b>Defining Probability Measures for Time Series .....</b>                           | <b>69</b> |
| 3.1      | Finite Dimensional Distributions .....                                               | 69        |
| 3.2      | Transformations and Equations.....                                                   | 70        |
| 3.3      | Conditions on the Expected Value .....                                               | 71        |
| 3.4      | Conditions on the Autocovariance Function .....                                      | 73        |
| 3.4.1    | Positive Semidefinite Functions .....                                                | 73        |
| 3.4.2    | Spectral Distribution .....                                                          | 77        |
| 3.4.3    | Calculation and Properties of $F$ and $f$ .....                                      | 86        |

|          |                                                                  |     |
|----------|------------------------------------------------------------------|-----|
| <b>4</b> | <b>Spectral Representation of Univariate Time Series</b>         | 101 |
| 4.1      | Motivation                                                       | 101 |
| 4.2      | Harmonic Processes                                               | 102 |
| 4.3      | Extension to General Processes                                   | 105 |
| 4.3.1    | Stochastic Integrals with Respect to $Z$                         | 105 |
| 4.3.2    | Existence and Definition of $Z$                                  | 112 |
| 4.3.3    | Interpretation of the Spectral Representation                    | 122 |
| 4.4      | Further Properties                                               | 122 |
| 4.4.1    | Relationship Between $\text{Re } Z$ and $\text{Im } Z$           | 122 |
| 4.4.2    | Frequency                                                        | 123 |
| 4.4.3    | Overtones                                                        | 124 |
| 4.4.4    | Why Are Frequencies Restricted to the Range $[-\pi, \pi]$ ? ...  | 125 |
| 4.5      | Linear Filters and the Spectral Representation                   | 129 |
| 4.5.1    | Effect on the Spectral Representation                            | 129 |
| 4.5.2    | Elimination of Frequency Bands                                   | 134 |
| <b>5</b> | <b>Spectral Representation of Real Valued Vector Time Series</b> | 137 |
| 5.1      | Cross-Spectrum and Spectral Representation                       | 137 |
| 5.2      | Coherence and Phase                                              | 146 |
| <b>6</b> | <b>Univariate ARMA Processes</b>                                 | 161 |
| 6.1      | Definition                                                       | 161 |
| 6.2      | Stationary Solution                                              | 161 |
| 6.3      | Causal Stationary Solution                                       | 166 |
| 6.4      | Causal Invertible Stationary Solution                            | 169 |
| 6.5      | Autocovariances of ARMA Processes                                | 170 |
| 6.5.1    | Calculation by Integration                                       | 170 |
| 6.5.2    | Calculation Using the Autocovariance Generating Function         | 170 |
| 6.5.3    | Calculation Using the Wold Representation                        | 175 |
| 6.5.4    | Recursive Calculation                                            | 176 |
| 6.5.5    | Asymptotic Decay                                                 | 177 |
| 6.6      | Integrated, Seasonal and Fractional ARMA and ARIMA Processes     | 185 |
| 6.6.1    | Integrated Processes                                             | 185 |
| 6.6.2    | Seasonal ARMA Processes                                          | 186 |
| 6.6.3    | Fractional ARIMA Processes                                       | 187 |
| 6.7      | Unit Roots, Spurious Correlation, Cointegration                  | 200 |
| <b>7</b> | <b>Generalized Autoregressive Processes</b>                      | 203 |
| 7.1      | Definition of Generalized Autoregressive Processes               | 203 |
| 7.2      | Stationary Solution of Generalized Autoregressive Equations      | 204 |
| 7.3      | Definition of VARMA Processes                                    | 209 |
| 7.4      | Stationary Solution of VARMA Equations                           | 211 |
| 7.5      | Definition of GARCH Processes                                    | 213 |
| 7.6      | Stationary Solution of GARCH Equations                           | 214 |

|           |                                                                                     |     |
|-----------|-------------------------------------------------------------------------------------|-----|
| 7.7       | Definition of ARCH( $\infty$ ) Processes .....                                      | 219 |
| 7.8       | Stationary Solution of ARCH( $\infty$ ) Equations .....                             | 220 |
| <b>8</b>  | <b>Prediction</b> .....                                                             | 223 |
| 8.1       | Best Linear Prediction Given an Infinite Past .....                                 | 223 |
| 8.2       | Predictability .....                                                                | 225 |
| 8.3       | Construction of the Wold Decomposition from $f$ .....                               | 230 |
| 8.4       | Best Linear Prediction Given a Finite Past .....                                    | 235 |
| <b>9</b>  | <b>Inference for <math>\mu</math>, <math>\gamma</math> and <math>F</math></b> ..... | 241 |
| 9.1       | Location Estimation .....                                                           | 241 |
| 9.2       | Linear Regression .....                                                             | 244 |
| 9.3       | Nonparametric Estimation of $\gamma$ .....                                          | 253 |
| 9.4       | Nonparametric Estimation of $f$ .....                                               | 262 |
| <b>10</b> | <b>Parametric Estimation</b> .....                                                  | 281 |
| 10.1      | Gaussian and Quasi Maximum Likelihood Estimation .....                              | 281 |
| 10.2      | Whittle Approximation .....                                                         | 284 |
| 10.3      | Autoregressive Approximation .....                                                  | 287 |
| 10.4      | Model Choice .....                                                                  | 289 |
|           | <b>References</b> .....                                                             | 293 |
|           | <b>Author Index</b> .....                                                           | 299 |
|           | <b>Subject Index</b> .....                                                          | 303 |