

Singular Elliptic Problems provides a comprehensive introduction to the mathematical theory of nonlinear problems described by singular elliptic equations. The book is an elementary introduction to nonlinear elliptic partial differential equations, which arise in the research of various nonlinear and singular phenomena within mathematical physics. Interest in this direction of research relates to the application of these problems in physics, quantum physics, mechanics, genetics, engineering, economics, and differential geometry.

The book lies at the interface between nonlinear analysis, mathematical physics, and variational calculus and is intended for graduate students, post-graduate students, and researchers in PDE and Pure and Applied Mathematics. It is the first monograph treating such nonlinear singular problems and develops two basic methods in nonlinear functional analysis: the maximum principle and critical point theorems. Some of the singular phenomena described in this book include existence (or nonexistence) of solutions, unicity or multiplicity problems, bifurcation and asymptotic analysis, and optimal regularity.

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