

Invited lecture courses

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EXTRAPOLATION AND FACTORIZATION

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1. INTRODUCTION

The purpose of these lecture notes is to give an overview of the theories of factorization and extrapolation for Muckenhoupt A_p weights. The A_p weights were introduced by Muckenhoupt [73] in the early 1970s and a wide ranging theory quickly developed: see [41, 46, 51] for details of this early history and extensive references.

Very early on the fine structure of A_p weights—e.g. the A_∞ condition, the reverse Hölder inequality and the fact that A_p implies A_{p-c} —began to play an important role in the theory. They were central to the proofs of the boundedness of maximal operators and singular integral operators on weighted spaces: see Coifman and Fefferman [13].

The deep structure revealed by the Jones factorization theorem—that every A_p weight can be factored as the product of two A_1 weights—was conjectured by Muckenhoupt [74] at the Williamstown conference in 1979, and Jones [60] proved it at

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