

Contents

Preface	v
0.1 Acknowledgements	viii
1 Introduction and Overview	1
1.1 Introduction	1
1.2 Basic principles of quantum mechanics	3
1.2.1 Fundamental axioms of quantum mechanics	3
1.2.2 Principles of measurement theory	5
1.3 Unitary group representations	6
1.3.1 Lie groups	7
1.3.2 Group representations	7
1.3.3 Unitary group representations	10
1.4 Representations and quantum mechanics	11
1.5 Groups and symmetries	12
1.6 For further reading	13
2 The Group $U(1)$ and its Representations	15
2.1 Some representation theory	16
2.2 The group $U(1)$ and its representations	19
2.3 The charge operator	21
2.4 Conservation of charge and $U(1)$ symmetry	24
2.5 Summary	25
2.6 For further reading	26
3 Two-state Systems and $SU(2)$	27
3.1 The two-state quantum system	28
3.1.1 The Pauli matrices: observables of the two-state quantum system	28
3.1.2 Exponentials of Pauli matrices: unitary transformations of the two-state system	30

3.2	Commutation relations for Pauli matrices	34
3.3	Dynamics of a two-state system	36
3.4	For further reading	37
4	Linear Algebra Review, Unitary and Orthogonal Groups	39
4.1	Vector spaces and linear maps	39
4.2	Dual vector spaces	41
4.3	Change of basis	43
4.4	Inner products	43
4.5	Adjoint operators	47
4.6	Orthogonal and unitary transformations	48
4.6.1	Orthogonal groups	48
4.6.2	Unitary groups	50
4.7	Eigenvalues and eigenvectors	51
4.8	For further reading	53
5	Lie Algebras and Lie Algebra Representations	55
5.1	Lie algebras	56
5.2	Lie algebras of the orthogonal and unitary groups	59
5.2.1	Lie algebra of the orthogonal group	60
5.2.2	Lie algebra of the unitary group	61
5.3	A summary	62
5.4	Lie algebra representations	63
5.5	Complexification	68
5.6	For further reading	71
6	The Rotation and Spin Groups in Three and Four Dimensions	73
6.1	The rotation group in three dimensions	73
6.2	Spin groups in three and four dimensions	76
6.2.1	Quaternions	77
6.2.2	Rotations and spin groups in four dimensions	78
6.2.3	Rotations and spin groups in three dimensions	79
6.2.4	The spin group and $SU(2)$	83
6.3	A summary	85
6.4	For further reading	86
7	Rotations and the Spin $\frac{1}{2}$ Particle in a Magnetic Field	87
7.1	The spinor representation	87
7.2	The spin $\frac{1}{2}$ particle in a magnetic field	89
7.3	The Heisenberg picture	93
7.4	Complex projective space	94
7.5	The Bloch sphere	98
7.6	For further reading	101

8	Representations of $SU(2)$ and $SO(3)$	103
8.1	Representations of $SU(2)$: classification	104
8.1.1	Weight decomposition	104
8.1.2	Lie algebra representations: raising and lowering operators	106
8.2	Representations of $SU(2)$: construction	112
8.3	Representations of $SO(3)$ and spherical harmonics	115
8.4	The Casimir operator	122
8.5	For further reading	124
9	Tensor Products, Entanglement, and Addition of Spin	125
9.1	Tensor products	126
9.2	Composite quantum systems and tensor products	128
9.3	Indecomposable vectors and entanglement	129
9.4	Tensor products of representations	130
9.4.1	Tensor products of $SU(2)$ representations	131
9.4.2	Characters of representations	132
9.4.3	Some examples	133
9.5	Bilinear forms and tensor products	135
9.6	Symmetric and antisymmetric multilinear forms	136
9.7	For further reading	138
10	Momentum and the Free Particle	139
10.1	The group $\mathbf{R}$ and its representations	140
10.2	Translations in time and space	142
10.2.1	Energy and the group $\mathbf{R}$ of time translations	142
10.2.2	Momentum and the group $\mathbf{R}^3$ of space translations	143
10.3	The energy-momentum relation and the Schrödinger equation for a free particle	144
10.4	For further reading	147
11	Fourier Analysis and the Free Particle	149
11.1	Periodic boundary conditions and the group $U(1)$	150
11.2	The group $\mathbf{R}$ and the Fourier transform	154
11.3	Distributions	157
11.4	Linear transformations and distributions	159
11.5	Solutions of the Schrödinger equation in momentum space	161
11.6	For further reading	164
12	Position and the Free Particle	165
12.1	The position operator	165
12.2	Momentum space representation	166
12.3	Dirac notation	167

12.4	Heisenberg uncertainty	169
12.5	The propagator in position space	170
12.6	Propagators in frequency–momentum space	174
12.7	Green’s functions and solutions to the Schrödinger equations	177
12.8	For further reading	180
13	The Heisenberg group and the Schrödinger Representation	181
13.1	The Heisenberg Lie algebra	182
13.2	The Heisenberg group	183
13.3	The Schrödinger representation	185
13.4	For further reading	188
14	The Poisson Bracket and Symplectic Geometry	189
14.1	Classical mechanics and the Poisson bracket	190
14.2	The Poisson bracket and the Heisenberg Lie algebra	192
14.3	Symplectic geometry	195
14.4	For further reading	198
15	Hamiltonian Vector Fields and the Moment Map	199
15.1	Vector fields and the exponential map	200
15.2	Hamiltonian vector fields and canonical transformations	202
15.3	Group actions on M and the moment map	207
15.4	Examples of Hamiltonian group actions	210
15.5	The dual of a Lie algebra and symplectic geometry	211
15.6	For further reading	214
16	Quadratic Polynomials and the Symplectic Group	215
16.1	The symplectic group	216
16.1.1	The symplectic group for $d = 1$	216
16.1.2	The symplectic group for arbitrary d	220
16.2	The symplectic group and automorphisms of the Heisenberg group	222
16.2.1	The adjoint representation and inner automorphisms	223
16.2.2	The symplectic group as automorphism group	224
16.3	The case of arbitrary d	226
16.4	For further reading	227
17	Quantization	229
17.1	Canonical quantization	229
17.2	The Groenewold–van Hove no-go theorem	232
17.3	Canonical quantization in d dimensions	234
17.4	Quantization and symmetries	234
17.5	More general notions of quantization	235
17.6	For further reading	236

18	Semi-direct Products	237
18.1	An example: the Euclidean group	237
18.2	Semi-direct product groups	239
18.3	Semi-direct product Lie algebras	241
18.4	For further reading	243
19	The Quantum Free Particle as a Representation of the Euclidean Group	245
19.1	The quantum free particle and representations of $E(2)$	246
19.2	The case of $E(3)$	251
19.3	Other representations of $E(3)$	254
19.4	For further reading	256
20	Representations of Semi-direct Products	259
20.1	Intertwining operators and the metaplectic representation	260
20.2	Constructing intertwining operators	262
20.3	Explicit calculations	263
20.3.1	The $SO(2)$ action by rotations of the plane for $d = 2$	263
20.3.2	An $SO(2)$ action on the $d = 1$ phase space	266
20.3.3	The Fourier transform as an intertwining operator	268
20.3.4	An $\mathbf{R}$ action on the $d = 1$ phase space	268
20.4	Representations of $N \rtimes K$, N commutative	269
20.5	For further reading	272
21	Central Potentials and the Hydrogen Atom	275
21.1	Quantum particle in a central potential	275
21.2	$\mathfrak{so}(4)$ symmetry and the Coulomb potential	280
21.3	The hydrogen atom	284
21.4	For further reading	285
22	The Harmonic Oscillator	287
22.1	The harmonic oscillator with one degree of freedom	288
22.2	Creation and annihilation operators	290
22.3	The Bargmann–Fock representation	294
22.4	Quantization by annihilation and creation operators	297
22.5	For further reading	298

23	Coherent States and the Propagator for the Harmonic Oscillator	299
23.1	Coherent states and the Heisenberg group action	299
23.2	Coherent states and the Bargmann–Fock state space	302
23.3	The Heisenberg group action on operators	305
23.4	The harmonic oscillator propagator	305
23.4.1	The propagator in the Bargmann–Fock representation	306
23.4.2	The coherent state propagator	307
23.4.3	The position space propagator	308
23.5	The Bargmann transform	309
23.6	For further reading	311
24	The Metaplectic Representation and Annihilation and Creation Operators, $d = 1$	313
24.1	The metaplectic representation for $d = 1$ in terms of a and $a^\dagger$	313
24.2	Intertwining operators in terms of a and $a^\dagger$	317
24.3	Implications of the choice of $z, \bar{z}$	320
24.4	$SU(1, 1)$ and Bogoliubov transformations	322
24.5	For further reading	324
25	The Metaplectic Representation and Annihilation and Creation Operators, arbitrary d	325
25.1	Multiple degrees of freedom	326
25.2	Complex coordinates on phase space and $U(d) \subset Sp(2d, \mathbf{R})$	327
25.3	The metaplectic representation and $U(d) \subset Sp(2d, \mathbf{R})$	331
25.4	Examples in $d = 2$ and 3	333
25.4.1	Two degrees of freedom and $SU(2)$	333
25.4.2	Three degrees of freedom and $SO(3)$	336
25.5	Normal ordering and the anomaly in finite dimensions	338
25.6	For further reading	339
26	Complex Structures and Quantization	341
26.1	Complex structures and phase space	341
26.2	Compatible complex structures and positivity	344
26.3	Complex structures and quantization	347
26.4	Complex vector spaces with Hermitian inner product as phase spaces	350
26.5	Complex structures for $d = 1$ and squeezed states	353
26.6	Complex structures and Bargmann–Fock quantization for arbitrary d	356
26.7	For further reading	356

27	The Fermionic Oscillator	357
27.1	Canonical anticommutation relations and the fermionic oscillator	357
27.2	Multiple degrees of freedom	359
27.3	For further reading	362
28	Weyl and Clifford Algebras	365
28.1	The Complex Weyl and Clifford algebras	365
28.1.1	One degree of freedom, bosonic case	365
28.1.2	One degree of freedom, fermionic case	367
28.1.3	Multiple degrees of freedom	368
28.2	Real Clifford algebras	370
28.3	For further reading	372
29	Clifford Algebras and Geometry	373
29.1	Non-degenerate bilinear forms	373
29.2	Clifford algebras and geometry	375
29.2.1	Rotations as iterated orthogonal reflections	377
29.2.2	The Lie algebra of the rotation group and quadratic elements of the Clifford algebra	379
29.3	For further reading	381
30	Anticommuting Variables and Pseudo-classical Mechanics	383
30.1	The Grassmann algebra of polynomials on anticommuting generators	383
30.2	Pseudo-classical mechanics and the fermionic Poisson bracket	386
30.3	Examples of pseudo-classical mechanics	390
30.3.1	The pseudo-classical spin degree of freedom	390
30.3.2	The pseudo-classical fermionic oscillator	391
30.4	For further reading	393
31	Fermionic Quantization and Spinors	395
31.1	Quantization of pseudo-classical systems	395
31.1.1	Quantization of the pseudo-classical spin	399
31.2	The Schrödinger representation for fermions: ghosts	400
31.3	Spinors and the Bargmann–Fock construction	402
31.4	Complex structures, $U(d) \subset SO(2d)$ and the spinor representation	405
31.5	An example: spinors for $SO(4)$	408
31.6	For further reading	410

32	A Summary: Parallels Between Bosonic and Fermionic Quantization	411
33	Supersymmetry, Some Simple Examples	413
33.1	The supersymmetric oscillator	413
33.2	Supersymmetric quantum mechanics with a superpotential	416
33.3	Supersymmetric quantum mechanics and differential forms	419
33.4	For further reading	420
34	The Pauli Equation and the Dirac Operator	421
34.1	The Pauli-Schrödinger equation and free spin $\frac{1}{2}$ particles in $d = 3$	421
34.2	Solutions of the Pauli equation and representations of $\widetilde{E(3)}$	424
34.3	The $E(3)$ -invariant inner product	429
34.4	The Dirac operator	430
34.5	For further reading	431
35	Lagrangian Methods and the Path Integral	433
35.1	Lagrangian mechanics	433
35.2	Noether's theorem and symmetries in the Lagrangian formalism	438
35.3	Quantization and path integrals	439
35.4	Advantages and disadvantages of the path integral	444
35.5	For further reading	445
36	Multiparticle Systems: Momentum Space Description	447
36.1	Multiparticle quantum systems as quanta of a harmonic oscillator	448
36.1.1	Bosons and the quantum harmonic oscillator	448
36.1.2	Fermions and the fermionic oscillator	450
36.2	Multiparticle quantum systems of free particles: finite cutoff formalism	452
36.3	Continuum formalism	456
36.4	Multiparticle wavefunctions	461
36.5	Dynamics	461
36.6	For further reading	463
37	Multiparticle Systems and Field Quantization	465
37.1	Quantum field operators	466
37.2	Quadratic operators and dynamics	468
37.3	The propagator in non-relativistic quantum field theory	471

37.4	Interacting quantum fields	472
37.5	Fermion fields	474
37.6	For further reading	475
38	Symmetries and Non-relativistic Quantum Fields	477
38.1	Unitary transformations on $\mathcal{H}_1$	478
38.2	Internal symmetries	479
38.2.1	$U(1)$ symmetry	479
38.2.2	$U(n)$ symmetry	481
38.3	Spatial symmetries	484
38.3.1	Spatial translations	485
38.3.2	Spatial rotations	487
38.3.3	Spin $\frac{1}{2}$ fields	488
38.4	Fermionic fields	489
38.5	For further reading	491
39	Quantization of Infinite dimensional Phase Spaces	493
39.1	Inequivalent irreducible representations	494
39.2	The restricted symplectic group	496
39.3	The anomaly and the Schwinger term	498
39.4	Spontaneous symmetry breaking	499
39.5	Higher-order operators and renormalization	500
39.6	For further reading	501
40	Minkowski Space and the Lorentz Group	503
40.1	Minkowski space	504
40.2	The Lorentz group and its Lie algebra	506
40.3	The Fourier transform in Minkowski space	509
40.4	Spin and the Lorentz group	510
40.5	For further reading	513
41	Representations of the Lorentz Group	515
41.1	Representations of the Lorentz group	515
41.2	Dirac γ -matrices and $\text{Cliff}(3, 1)$	521
41.3	For further reading	525
42	The Poincaré Group and its Representations	527
42.1	The Poincaré group and its Lie algebra	528
42.2	Irreducible representations of the Poincaré group	530
42.3	Classification of representations by orbits	534
42.3.1	Positive energy time-like orbits	535
42.3.2	Negative energy time-like orbits	536
42.3.3	Space-like orbits	537
42.3.4	The zero orbit	537
42.3.5	Positive energy null orbits	537
42.3.6	Negative energy null orbits	539
42.4	For further reading	539

43	The Klein–Gordon Equation and Scalar Quantum Fields	541
43.1	The Klein–Gordon equation and its solutions	541
43.2	The symplectic and complex structures on $\mathcal{M}$	547
43.3	Hamiltonian and dynamics of the Klein–Gordon theory	551
43.4	Quantization of the Klein–Gordon theory	553
43.5	The scalar field propagator	556
43.6	Interacting scalar field theories: some comments	558
43.7	For further reading	559
44	Symmetries and Relativistic Scalar Quantum Fields	561
44.1	Internal symmetries	562
44.1.1	$SO(m)$ symmetry and real scalar fields	562
44.1.2	$U(1)$ symmetry and complex scalar fields	565
44.2	Poincaré symmetry and scalar fields	568
44.2.1	Translations	570
44.2.2	Rotations	571
44.2.3	Boosts	571
44.3	For further reading	572
45	$U(1)$ Gauge Symmetry and Electromagnetic Fields	573
45.1	$U(1)$ gauge symmetry	573
45.2	Curvature, electric and magnetic fields	576
45.3	Field equations with background electromagnetic fields	577
45.4	The geometric significance of the connection	579
45.5	The non-Abelian case	581
45.6	For further reading	583
46	Quantization of the Electromagnetic Field: the Photon	585
46.1	Maxwell’s equations	585
46.2	The Hamiltonian formalism for electromagnetic fields	587
46.3	Gauss’s law and time-independent gauge transformations	589
46.4	Quantization in Coulomb gauge	592
46.5	Space–time symmetries	595
46.5.1	Time translations	596
46.5.2	Spatial translations	596
46.5.3	Rotations	597
46.6	Covariant gauge quantization	598
46.7	For further reading	603

47	The Dirac Equation and Spin $\frac{1}{2}$ Fields	605
47.1	The Dirac equation in Minkowski space	605
47.2	Majorana spinors and the Majorana field	609
47.2.1	Majorana spinor fields in momentum space	611
47.2.2	Quantization of the Majorana field	614
47.3	Weyl spinors	615
47.4	Dirac spinors	619
47.5	For further reading	620
48	An Introduction to the Standard Model	621
48.1	Non-Abelian gauge fields	621
48.2	Fundamental fermions	622
48.3	Spontaneous symmetry breaking	623
48.4	Unanswered questions and speculative extensions	623
48.4.1	Why these gauge groups and couplings?	624
48.4.2	Why these representations?	624
48.4.3	Why three generations?	624
48.4.4	Why the Higgs field?	625
48.4.5	Why the Yukawas?	625
48.4.6	What is the dynamics of the gravitational field?	625
48.5	For further reading	626
49	Further Topics	627
49.1	Connecting quantum theories to experimental results	628
49.2	Other important mathematical physics topics	628
A	Conventions	631
A.1	Bilinear forms	631
A.2	Fourier transforms	632
A.3	Symplectic geometry and quantization	632
A.4	Complex structures and Bargmann-Fock quantization	634
A.5	Special relativity	635
A.6	Clifford algebras and spinors	635
B	Exercises	637
B.1	Chapters 1 and 2	637
B.2	Chapters 3 and 4	638
B.3	Chapters 5 to 7	639
B.4	Chapter 8	641
B.5	Chapter 9	642
B.6	Chapters 10 to 12	643
B.7	Chapters 14 to 16	644
B.8	Chapter 17	645
B.9	Chapters 18 and 19	646
B.10	Chapters 21 and 22	647
B.11	Chapter 23	649

B.12	Chapters 24 to 26	649
B.13	Chapters 27 and 28	650
B.14	Chapters 29 to 31	651
B.15	Chapters 33 and 34	653
B.16	Chapter 36	653
B.17	Chapters 37 and 38	654
B.18	Chapters 40 to 42	655
B.19	Chapters 43 and 44	656
B.20	Chapters 45 and 46	656
B.21	Chapter 47	657
Bibliography		659
Index		665