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*Gratitude is merely the secret hope of further favors.*

— François de La Rochefoucauld (1613–1680)

*So much time, and so little to do!*

*Strike that, reverse it.*

— Willy Wonka