

# Contents

|                                                           |      |
|-----------------------------------------------------------|------|
| Preface                                                   | vii  |
| Introduction                                              | xv   |
| About This Book                                           | xvii |
| CHAPTER I                                                 |      |
| Polish Spaces                                             | 1    |
| 1. Topological and Metric Spaces                          | 1    |
| 1.A Topological Spaces                                    | 1    |
| 1.B Metric Spaces                                         | 2    |
| 2. Trees                                                  | 5    |
| 2.A Basic Concepts                                        | 5    |
| 2.B Trees and Closed Sets                                 | 7    |
| 2.C Trees on Products                                     | 9    |
| 2.D Leftmost Branches                                     | 9    |
| 2.E Well-founded Trees and Ranks                          | 10   |
| 2.F The Well-founded Part of a Tree                       | 11   |
| 2.G The Kleene-Brouwer Ordering                           | 11   |
| 3. Polish Spaces                                          | 13   |
| 3.A Definitions and Examples                              | 13   |
| 3.B Extensions of Continuous Functions and Homeomorphisms | 15   |
| 3.C Polish Subspaces of Polish Spaces                     | 17   |
| 4. Compact Metrizable Spaces                              | 18   |
| 4.A Basic Facts                                           | 18   |
| 4.B Examples                                              | 19   |

|                                                              |           |
|--------------------------------------------------------------|-----------|
| 4.C A Universality Property of the Hilbert Cube              | 22        |
| 4.D Continuous Images of the Cantor Space                    | 23        |
| 4.E The Space of Continuous Functions on a Compact Space     | 24        |
| 4.F The Hyperspace of Compact Sets                           | 24        |
| <b>5. Locally Compact Spaces</b>                             | <b>29</b> |
| <b>6. Perfect Polish Spaces</b>                              | <b>31</b> |
| 6.A Embedding the Cantor Space in Perfect Polish Spaces      | 31        |
| 6.B The Cantor-Bendixson Theorem                             | 32        |
| 6.C Cantor-Bendixson Derivatives and Ranks                   | 33        |
| <b>7. Zero-dimensional Spaces</b>                            | <b>35</b> |
| 7.A Basic Facts                                              | 35        |
| 7.B A Topological Characterization of the Cantor Space       | 35        |
| 7.C A Topological Characterization of the Baire Space        | 36        |
| 7.D Zero-dimensional Spaces as Subspaces of the Baire Space  | 38        |
| 7.E Polish Spaces as Continuous Images of the Baire Space    | 38        |
| 7.F Closed Subsets Homeomorphic to the Baire Space           | 39        |
| <b>8. Baire Category</b>                                     | <b>41</b> |
| 8.A Meager Sets                                              | 41        |
| 8.B Baire Spaces                                             | 41        |
| 8.C Choquet Games and Spaces                                 | 43        |
| 8.D Strong Choquet Games and Spaces                          | 44        |
| 8.E A Characterization of Polish Spaces                      | 45        |
| 8.F Sets with the Baire Property                             | 47        |
| 8.G Localization                                             | 48        |
| 8.H The Banach-Mazur Game                                    | 51        |
| 8.I Baire Measurable Functions                               | 52        |
| 8.J Category Quantifiers                                     | 53        |
| 8.K The Kuratowski-Ulam Theorem                              | 53        |
| 8.L Some Applications                                        | 55        |
| 8.M Separate and Joint Continuity                            | 56        |
| <b>9. Polish Groups</b>                                      | <b>58</b> |
| 9.A Metrizable and Polish Groups                             | 58        |
| 9.B Examples of Polish Groups                                | 58        |
| 9.C Basic Facts about Baire Groups and Their Actions         | 60        |
| 9.D Universal Polish Groups                                  | 63        |
| <b>CHAPTER II</b>                                            |           |
| <b>Borel Sets</b>                                            | <b>65</b> |
| <b>10. Measurable Spaces and Functions</b>                   | <b>65</b> |
| 10.A Sigma-Algebras and Their Generators                     | 65        |
| 10.B Measurable Spaces and Functions                         | 66        |
| <b>11. Borel Sets and Functions</b>                          | <b>68</b> |
| 11.A Borel Sets in Topological Spaces                        | 68        |
| 11.B The Borel Hierarchy                                     | 68        |
| 11.C Borel Functions                                         | 70        |
| <b>12. Standard Borel Spaces</b>                             | <b>73</b> |
| 12.A Borel Sets and Functions in Separable Metrizable Spaces | 73        |
| 12.B Standard Borel Spaces                                   | 74        |

|                                                            |            |
|------------------------------------------------------------|------------|
| 12.C The Effros Borel Space                                | 75         |
| 12.D An Application to Selectors                           | 77         |
| 12.E Further Examples                                      | 78         |
| 12.F Standard Borel Groups                                 | 80         |
| <b>13. Borel Sets as Clopen Sets</b>                       | <b>82</b>  |
| 13.A Turning Borel into Clopen Sets                        | 82         |
| 13.B Other Representations of Borel Sets                   | 83         |
| 13.C Turning Borel into Continuous Functions               | 84         |
| <b>14. Analytic Sets and the Separation Theorem</b>        | <b>85</b>  |
| 14.A Basic Facts about Analytic Sets                       | 85         |
| 14.B The Lusin Separation Theorem                          | 87         |
| 14.C Souslin's Theorem                                     | 87         |
| <b>15. Borel Injections and Isomorphisms</b>               | <b>89</b>  |
| 15.A Borel Injective Images of Borel Sets                  | 89         |
| 15.B The Isomorphism Theorem                               | 90         |
| 15.C Homomorphisms of Sigma-Algebras Induced by Point Maps | 91         |
| 15.D Some Applications to Group Actions                    | 92         |
| <b>16. Borel Sets and Baire Category</b>                   | <b>94</b>  |
| 16.A Borel Definability of Category Notions                | 94         |
| 16.B The Vaught Transforms                                 | 95         |
| 16.C Connections with Model Theory                         | 96         |
| 16.D Connections with Cohen's Forcing Method               | 99         |
| <b>17. Borel Sets and Measures</b>                         | <b>103</b> |
| 17.A General Facts on Measures                             | 103        |
| 17.B Borel Measures                                        | 105        |
| 17.C Regularity and Tightness of Measures                  | 107        |
| 17.D Lusin's Theorem on Measurable Functions               | 108        |
| 17.E The Space of Probability Borel Measures               | 109        |
| 17.F The Isomorphism Theorem for Measures                  | 116        |
| <b>18. Uniformization Theorems</b>                         | <b>120</b> |
| 18.A The Jankov, von Neumann Uniformization Theorem        | 120        |
| 18.B "Large Section" Uniformization Results                | 122        |
| 18.C "Small Section" Uniformization Results                | 123        |
| 18.D Selectors and Transversals                            | 128        |
| <b>19. Partition Theorems</b>                              | <b>129</b> |
| 19.A Partitions with a Comeager or Non-meager Piece        | 129        |
| 19.B A Ramsey Theorem for Polish Spaces                    | 130        |
| 19.C The Galvin-Prikry Theorem                             | 132        |
| 19.D Ramsey Sets and the Ellentuck Topology                | 132        |
| 19.E An Application to Banach Space Theory                 | 134        |
| <b>20. Borel Determinacy</b>                               | <b>137</b> |
| 20.A Infinite Games                                        | 137        |
| 20.B Determinacy of Closed Games                           | 138        |
| 20.C Borel Determinacy                                     | 140        |
| 20.D Game Quantifiers                                      | 147        |
| <b>21. Games People Play</b>                               | <b>149</b> |
| 21.A The $*$ -Games                                        | 149        |
| 21.B Unfolding                                             | 150        |
| 21.C The Banach-Mazur or $**$ -Games                       | 151        |

|                                                          |            |
|----------------------------------------------------------|------------|
| 21.D The General Unfolded Banach-Mazur Games             | 153        |
| 21.E Wadge Games                                         | 156        |
| 21.F Separation Games and Hurewicz's Theorem             | 160        |
| 21.G Turing Degrees                                      | 164        |
| <b>22. The Borel Hierarchy</b>                           | <b>167</b> |
| 22.A Universal Sets                                      | 167        |
| 22.B The Borel versus the Wadge Hierarchy                | 169        |
| 22.C Structural Properties                               | 170        |
| 22.D Additional Results                                  | 173        |
| 22.E The Difference Hierarchy                            | 175        |
| <b>23. Some Examples</b>                                 | <b>179</b> |
| 23.A Combinatorial Examples                              | 179        |
| 23.B Classes of Compact Sets                             | 181        |
| 23.C Sequence Spaces                                     | 182        |
| 23.D Classes of Continuous Functions                     | 182        |
| 23.E Uniformly Convergent Sequences                      | 185        |
| 23.F Some Universal Sets                                 | 185        |
| 23.G Further Examples                                    | 188        |
| <b>24. The Baire Hierarchy</b>                           | <b>190</b> |
| 24.A The Baire Classes of Functions                      | 190        |
| 24.B Functions of Baire Class 1                          | 192        |
| <b>CHAPTER III</b>                                       |            |
| Analytic Sets                                            | 196        |
| <b>25. Representations of Analytic Sets</b>              | <b>196</b> |
| 25.A Review                                              | 196        |
| 25.B Analytic Sets in the Baire Space                    | 197        |
| 25.C The Souslin Operation                               | 198        |
| 25.D Wellordered Unions and Intersections of Borel Sets  | 201        |
| 25.E Analytic Sets as Open Sets in Strong Choquet Spaces | 202        |
| <b>26. Universal and Complete Sets</b>                   | <b>205</b> |
| 26.A Universal Analytic Sets                             | 205        |
| 26.B Analytic Determinacy                                | 205        |
| 26.C Complete Analytic Sets                              | 206        |
| 26.D Classification up to Borel Isomorphism              | 207        |
| <b>27. Examples</b>                                      | <b>209</b> |
| 27.A The Class of Ill-founded Trees                      | 209        |
| 27.B Classes of Closed Sets                              | 209        |
| 27.C Classes of Structures in Model Theory               | 212        |
| 27.D Isomorphism                                         | 213        |
| 27.E Some Universal Sets                                 | 214        |
| 27.F Miscellanea                                         | 215        |
| <b>28. Separation Theorems</b>                           | <b>217</b> |
| 28.A The Lusin Separation Theorem Revisited              | 217        |
| 28.B The Novikov Separation Theorem                      | 219        |
| 28.C Borel Sets with Open or Closed Sections             | 220        |
| 28.D Some Special Separation Theorems                    | 221        |
| 28.E "Hurewicz-Type" Separation Theorems                 | 224        |

|                                                                 |            |
|-----------------------------------------------------------------|------------|
| <b>29. Regularity Properties</b>                                | <b>226</b> |
| 29.A The Perfect Set Property                                   | 226        |
| 29.B Measure, Category, and Ramsey                              | 226        |
| 29.C A Closure Property for the Souslin Operation               | 227        |
| 29.D The Class of $C$ -Sets                                     | 230        |
| 29.E Analyticity of "Largeness" Conditions on Analytic Sets     | 230        |
| <b>30. Capacities</b>                                           | <b>234</b> |
| 30.A The Basic Concept                                          | 234        |
| 30.B Examples                                                   | 234        |
| 30.C The Choquet Capacitability Theorem                         | 237        |
| <b>31. Analytic Well-founded Relations</b>                      | <b>239</b> |
| 31.A Bounds on Ranks of Analytic Well-founded Relations         | 239        |
| 31.B The Kunen-Martin Theorem                                   | 241        |
| <br><b>CHAPTER IV</b>                                           |            |
| <b>Co-Analytic Sets</b>                                         | <b>242</b> |
| <b>32. Review</b>                                               | <b>242</b> |
| 32.A Basic Facts                                                | 242        |
| 32.B Representations of Co-Analytic Sets                        | 243        |
| 32.C Regularity Properties                                      | 244        |
| <b>33. Examples</b>                                             | <b>245</b> |
| 33.A Well-founded Trees and Wellorderings                       | 245        |
| 33.B Classes of Closed Sets                                     | 245        |
| 33.C Sigma-Ideals of Compact Sets                               | 246        |
| 33.D Differentiable Functions                                   | 248        |
| 33.E Everywhere Convergence                                     | 251        |
| 33.F Parametrizing Baire Class 1 Functions                      | 252        |
| 33.G A Method for Proving Completeness                          | 253        |
| 33.H Singular Functions                                         | 254        |
| 33.I Topological Examples                                       | 255        |
| 33.J Homeomorphisms of Compact Spaces                           | 257        |
| 33.K Classes of Separable Banach Spaces                         | 262        |
| 33.L Other Examples                                             | 266        |
| <b>34. Co-Analytic Ranks</b>                                    | <b>267</b> |
| 34.A Ranks and Prewellorderings                                 | 267        |
| 34.B Ranked Classes                                             | 267        |
| 34.C Co-Analytic Ranks                                          | 268        |
| 34.D Derivatives                                                | 270        |
| 34.E Co-Analytic Ranks Associated with Borel Derivatives        | 272        |
| 34.F Examples                                                   | 275        |
| <b>35. Rank Theory</b>                                          | <b>281</b> |
| 35.A Basic Properties of Ranked Classes                         | 281        |
| 35.B Parametrizing Bi-Analytic and Borel Sets                   | 283        |
| 35.C Reflection Theorems                                        | 285        |
| 35.D Boundedness Properties of Ranks                            | 288        |
| 35.E The Rank Method                                            | 290        |
| 35.F The Strategic Uniformization Theorem                       | 291        |
| 35.G Co-Analytic Families of Closed Sets and Their Sigma-Ideals | 292        |



|                                                         |            |
|---------------------------------------------------------|------------|
| 35.H Borel Sets with $F_\sigma$ and $K_\sigma$ Sections | 296        |
| <b>36. Scales and Uniformization</b>                    | <b>299</b> |
| 36.A Kappa-Souslin Sets                                 | 299        |
| 36.B Scales                                             | 299        |
| 36.C Scaled Classes and Uniformization                  | 302        |
| 36.D The Novikov-Kondô Uniformization Theorem           | 304        |
| 36.E Regularity Properties of Uniformizing Functions    | 307        |
| 36.F Uniformizing Co-Analytic Sets with Large Sections  | 309        |
| 36.G Examples of Co-Analytic Scales                     | 310        |
| <b>CHAPTER V</b>                                        |            |
| <b>Projective Sets</b>                                  | <b>313</b> |
| <b>37. The Projective Hierarchy</b>                     | <b>313</b> |
| 37.A Basic Facts                                        | 313        |
| 37.B Examples                                           | 316        |
| <b>38. Projective Determinacy</b>                       | <b>322</b> |
| 38.A The Second Level of the Projective Hierarchy       | 322        |
| 38.B Projective Determinacy                             | 325        |
| 38.C Regularity Properties                              | 326        |
| <b>39. The Periodicity Theorems</b>                     | <b>327</b> |
| 39.A Periodicity in the Projective Hierarchy            | 327        |
| 39.B The First Periodicity Theorem                      | 327        |
| 39.C The Second Periodicity Theorem                     | 336        |
| 39.D The Third Periodicity Theorem                      | 342        |
| <b>40. Epilogue</b>                                     | <b>346</b> |
| 40.A Extensions of the Projective Hierarchy             | 346        |
| 40.B Effective Descriptive Set Theory                   | 346        |
| 40.C Large Cardinals                                    | 346        |
| 40.D Connections to Other Areas of Mathematics          | 347        |
| Appendix A. Ordinals and Cardinals                      | 349        |
| Appendix B. Well-founded Relations                      | 351        |
| Appendix C. On Logical Notation                         | 353        |
| Notes and Hints                                         | 357        |
| References                                              | 369        |
| Symbols and Abbreviations                               | 381        |
| Index                                                   | 387        |