

LONDON MATHEMATICAL SOCIETY LECTURE NOTE SERIES

Edited by PROFESSOR I. M. JAMES
Mathematical Institute, 24-29 St Giles, Oxford

with the assistance of
F. E. Browder (*Chicago*)
M. W. Hirsch (*Berkeley*)
N. Katz (*Princeton*)
G.-C. Rota (*M.I.T.*)
S. T. Yau (*Stanford*)

Synthetic Differential Geometry

ANDERS KOCK
Matematisk Institut, Aarhus University

This is the first exposition of a synthetic method of reasoning in differential geometry and differential calculus, based on the assumption of sufficiently many nilpotent elements on the real line. The use of nilpotent elements allows one to replace the limit processes of analytic differential geometry and calculus by purely algebraic calculations. In the long run this feature may make those topics accessible at an earlier stage.

The use of nilpotent elements in geometry was advocated by Grothendieck but only when Lawvere, in 1967, studied them using categorical logic did their use become natural and intuitive.

Dr Kock assumes some knowledge of calculus and abstract algebra and familiarity with the basic notions of category theory. The graduate student or professional interested in algebraic or differential geometry, or category theory will find this book most useful, either as a textbook (there are numerous exercises) or as a reference book.

PrefacePart I: The Synthetic Theory

Introduction	1
1. Basic structure on the geometric line	2
2. Differential calculus	9
3. Higher Taylor formulae (one variable)	13
4. Partial derivatives	16
5. Higher Taylor formulae in several variables. Taylor series	21
6. Some important infinitesimal objects	25
7. Tangent vectors and the tangent bundle	33
8. Vector fields and infinitesimal transformations	39
9. Lie bracket - commutator of infinitesimal transformations	45
10. Directional derivatives	50
11. Some abstract algebra and functional analysis. Application to proof of Jacobi identity	57
12. The comprehensive axiom	61
13. Order and integration	69
14. Forms and currents	74
15. Currents defined using integration. Stokes' Theorem	83
16. Weil algebras	88
17. Formal manifolds	99
18. Differential forms in terms of 1-neighbour simplices	108
19. Open covers	117
20. Differential forms as quantities	124
21. Pure Geometry	129

PART II: Categorical Logic

Introduction	133
1. Generalized elements	135
2. Satisfaction (1)	138
3. Extensions and descriptions	144
4. Semantics of function objects	151
5. Axiom 1 revisited	158
6. Comma categories	161
7. Dense class of generators	169
8. Satisfaction (2), and topological density	173
9. Geometric theories	179

PART III: Models

Introduction	182
1. Models for Axioms 1, 2, and 3	183
2. Models for ε -stable geometric theories	192
3. Axiomatic theory of well-adapted models (1)	200
4. Axiomatic theory of well-adapted models (2)	208
5. The algebraic theory of smooth functions	216
6. Germ-determined \mathbb{T}_0 -algebras	229
7. The open cover topology	237
8. Construction of well-adapted models	244
9. W-determined algebras, and manifolds with boundary	252
10. A field property of R , and the synthetic role of germ algebras	267
11. Order and integration in the Cahiers topos	276
<u>Loose ends</u>	285
<u>Historical remarks</u>	288
<u>Appendix A: Functorial semantics</u>	295
<u>Appendix B: Grothendieck topologies</u>	300
<u>Appendix C: Cartesian closed categories</u>	303
<u>References</u>	304
<u>Index</u>	310