

Topology of Transitive Transformation Groups is devoted to homogeneous spaces of compact Lie groups or, which is the same, to transitive actions of compact Lie groups on smooth manifolds. The main problem discussed is that of classifying all transitive actions of compact Lie groups on a given homogeneous space. As a special case, the problem of determining all possible inclusions among compact transitive Lie transformation groups is considered. This is equivalent to the description of all factorizations of an arbitrary compact Lie group into a product of two Lie subgroups.

To solve these problems, the topological approach is used leading to the study of the topology of compact Lie groups and their homogeneous spaces. The book contains a detailed exposition of the real cohomology and the real homotopy theory of these spaces including H. Cartan's famous theorem.

To make this book as self-contained as possible, the author has included introductory sections about Lie groups, homogenous spaces, graded algebras, and cochain complexes.

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