

Categories, Bundles and Spacetime Topology

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The first part of this book presents an introductory course on category theory which is then applied to display the naturalness of several commonly occurring processes in topology. The middle part is devoted to manifolds, vector and fibre bundles and extra structure like connections, metrics and differential forms. Finally, much of the foregoing is brought to bear on the structure of spacetime manifolds. Here, the emphasis is on differential topological properties through a study of the existence of Lorentz structures, orientability, parallelizability, singularities and their stability. The combination of rigorous proofs, well-motivated constructions and many examples recommends this book as useful supplementary reading for courses in topology and differential geometry.

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