Contents

1	Ge	ometry and Complex Arithmetic					
	1	Introduction Historical Sketch Bombelli's "Wild Thought" Some Terminology and Notation Practice Symbolic and Geometric Arithmetic	1 1 3 6 7 8				
		Euler's Formula Introduction Moving Particle Argument Power Series Argument Sine and Cosine in Terms of Euler's Formula	10 10 10 12 14				
	111	Some Applications Introduction Trigonometry Geometry Calculus Algebra Vectorial Operations	14 14 14 16 20 22 27				
	IV	Transformations and Euclidean Geometry* Geometry Through the Eyes of Felix Klein Classifying Motions Three Reflections Theorem Similarities and Complex Arithmetic Spatial Complex Numbers?	30 30 34 37 39 43				
	٧	Exercises	45				
2	Co	mplex Functions as Transformations	55				
	1	Introduction	55				
	11	Polynomials 1 Positive Integer Powers 2 Cubics Revisited* 3 Cassinian Curves*	57 57 59 60				
	Ш	Power Series The Mystery of Real Power Series The Disc of Convergence	64 64 67				

xvi Contents

		 Approximating a Power Series with a Polynomial Uniqueness Manipulating Power Series Finding the Radius of Convergence Fourier Series* 	70 71 72 74 77
	IV	The Exponential Function Power Series Approach The Geometry of the Mapping Another Approach	79 79 80 81
	V	Cosine and Sine Definitions and Identities Relation to Hyperbolic Functions The Geometry of the Mapping	84 84 86 88
	VI	Multifunctions Example: Fractional Powers Single-Valued Branches of a Multifunction Relevance to Power Series An Example with Two Branch Points	90 90 92 95 96
	VII	The Logarithm Function Inverse of the Exponential Function The Logarithmic Power Series General Powers	98 98 100 101
	VIII	Averaging over Circles* 1 The Centroid 2 Averaging over Regular Polygons 3 Averaging over Circles	102 102 105 108
	IX	Exercises	111
3	Möl	oius Transformations and Inversion	122
	I	Introduction 1 Definition of Möbius Transformations 2 Connection with Einstein's Theory of Relativity* 3 Decomposition into Simple Transformations	122 122 122 123
	11	Inversion Preliminary Definitions and Facts Preservation of Circles Construction Using Orthogonal Circles Preservation of Angles Preservation of Symmetry Inversion in a Sphere	124 124 126 128 130 133
	111	Three Illustrative Applications of Inversion 1 A Problem on Touching Circles 2 Quadrilaterals with Orthogonal Diagonals 3 Ptolemy's Theorem	136 136 137 138

			Contents	xvii
	IV	The Riemann Sphere 1 The Point at Infinity 2 Stereographic Projection 3 Transferring Complex Functions to the Sphere 4 Behaviour of Functions at Infinity 5 Stereographic Formulae*		139 139 140 143 144 146
	V	Möbius Transformations: Basic Results 1 Preservation of Circles, Angles, and Symmetry 2 Non-Uniqueness of the Coefficients 3 The Group Property 4 Fixed Points 5 Fixed Points at Infinity 6 The Cross-Ratio		148 149 150 151 152 154
	VI	Möbius Transformations as Matrices* Evidence of a Link with Linear Algebra The Explanation: Homogeneous Coordinates Eigenvectors and Eigenvalues* Rotations of the Sphere*		156 156 157 158 161
	VII	Visualization and Classification* The Main Idea Elliptic, Hyperbolic, and Loxodromic types Local Geometric Interpretation of the Multiplier Parabolic Transformations Computing the Multiplier* Eigenvalue Interpretation of the Multiplier*		162 164 166 168 169 170
	VIII	Decomposition into 2 or 4 Reflections* Introduction Elliptic Case Hyperbolic Case Parabolic Case Summary		172 172 172 173 174 175
	IX	Automorphisms of the Unit Disc* Counting Degrees of Freedom Finding the Formula via the Symmetry Principle Interpreting the Formula Geometrically* Introduction to Riemann's Mapping Theorem		176 176 177 178 180
	Х	Exercises		181
4	Diff	erentiation: The Amplitwist Concept		189
	1	Introduction		189
	II	A Puzzling Phenomenon		189
	111	Local Description of Mappings in the Plane Introduction The Jacobian Matrix The Amplitwist Concept		191 191 192 193

xviii Contents

	IV	The Complex Derivative as Amplitwist The Real Derivative Re-examined The Complex Derivative Analytic Functions A Brief Summary	194 194 195 197 198
	V	Some Simple Examples	199
	VI	Conformal = Analytic Introduction Conformality Throughout a Region Conformality and the Riemann Sphere	200 200 201 203
	VII	Critical Points Degrees of Crushing Breakdown of Conformality Branch Points	204 204 205 206
	VIII	The Cauchy-Riemann Equations Introduction The Geometry of Linear Transformations The Cauchy-Riemann Equations	207 207 208 209
	IX	Exercises	211
5	Fur	ther Geometry of Differentiation	216
	1	Cauchy-Riemann Revealed Introduction The Cartesian Form The Polar Form	216 216 216 217
	II	An Intimation of Rigidity	219
		Visual Differentiation of log(z)	222
	IV	Rules of Differentiation Composition Inverse Functions Addition and Multiplication	223 223 224 225
	V	Polynomials, Power Series, and Rational Functions Polynomials Power Series Rational Functions	226 226 227 228
	VI	Visual Differentiation of the Power Function	229
	VII	Visual Differentiation of exp(z)	231
	VIII	Geometric Solution of E' = E	232
	IX	An Application of Higher Derivatives: Curvature* 1 Introduction	234 234

				Contents	xix
		2 3	Analytic Transformation of Curvature Complex Curvature		235 238
	X	Celes 1 2 3 4 5 6	Central Force Fields Two Kinds of Elliptical Orbit Changing the First into the Second The Geometry of Force An Explanation The Kasner–Arnol'd Theorem		241 241 241 243 244 245 246
	XI	Analy 1 2 3 4 5	rtic Continuation* Introduction Rigidity Uniqueness Preservation of Identities Analytic Continuation via Reflections		247 247 249 250 251 252
	XII	Exerc	cises		258
6	Nor	-Eucl	idean Geometry*		267
		Introd 1 2 3 4 5 6 7	The Parallel Axiom Some Facts from Non-Euclidean Geometry Geometry on a Curved Surface Intrinsic versus Extrinsic Geometry Gaussian Curvature Surfaces of Constant Curvature The Connection with Möbius Transformations		267 267 269 270 273 273 275 277
	II	Sphe 1 2 3 4 5	The Angular Excess of a Spherical Triangle Motions of the Sphere A Conformal Map of the Sphere Spatial Rotations as Möbius Transformations Spatial Rotations and Quaternions		278 278 279 283 286 290
	III	Hype 1 2 3 4 5 6 7 8 9 10 11 12 Exerc	The Tractrix and the Pseudosphere The Constant Curvature of the Pseudosphere* A Conformal Map of the Pseudosphere Beltrami's Hyperbolic Plane Hyperbolic Lines and Reflections The Bolyai-Lobachevsky Formula* The Three Types of Direct Motion Decomposition into Two Reflections The Angular Excess of a Hyperbolic Triangle The Poincaré Disc Motions of the Poincaré Disc The Hemisphere Model and Hyperbolic Space		293 293 295 296 298 301 305 306 311 313 315 319 322
	IV	Exer	cises		328

7	Win	ding Numbers and Topology	338
	1	Winding Number The Definition What does "inside" mean? Finding Winding Numbers Quickly	338 338 339 340
	11	Hopf's Degree Theorem 1 The Result 2 Loops as Mappings of the Circle* 3 The Explanation*	341 341 342 343
	Ш	Polynomials and the Argument Principle	344
	IV	A Topological Argument Principle* Counting Preimages Algebraically Counting Preimages Geometrically Topological Characteristics of Analyticity A Topological Argument Principle Two Examples	346 346 347 349 350 352
	V	Rouché's Theorem The Result The Fundamental Theorem of Algebra Brouwer's Fixed Point Theorem*	353 353 354 354
	VI	Maxima and Minima Maximum-Modulus Theorem Related Results	355 355 357
	VII	The Schwarz-Pick Lemma* 1 Schwarz's Lemma 2 Liouville's Theorem 3 Pick's Result	357 357 359 360
	VIII	The Generalized Argument Principle Rational Functions Poles and Essential Singularities The Explanation*	363 363 365 367
	IX	Exercises	369
8	Cor	mplex Integration: Cauchy's Theorem	377
	1	Introduction	377
	11	The Real Integral The Riemann Sum The Trapezoidal Rule Geometric Estimation of Errors	378 378 379 380
	111	The Complex Integral Complex Riemann Sums A Visual Technique A Useful Inequality	383 383 386 386

		Contents	xxi
	4 Rules of Integration		387
IV	Complex Inversion		388
	1 A Circular Arc 2 General Loops		388
	3 Winding Number		391
V	Conjugation		392
	1 Introduction		392
	2 Area Interpretation 3 General Loops		393 395
VI	Power Functions		395
	1 Integration along a Circular Arc		395
	 Complex Inversion as a Limiting Case* General Contours and the Deformation Theorem 		397
	 General Contours and the Deformation Theorem A Further Extension of the Theorem 		397 399
	5 Residues		400
VII	The Exponential Mapping		401
VIII	The Fundamental Theorem		402
	1 Introduction2 An Example		402 403
	3 The Fundamental Theorem		404
A	The Integral as Antiderivative		406
IV	5 Logarithm as Integral		408
IX	Parametric Evaluation		409
X	Cauchy's Theorem Some Preliminaries		410 410
	2 The Explanation		412
XI	The General Cauchy Theorem		414
	1 The Result		414
	The ExplanationA Simpler Explanation		415 417
XII	The General Formula of Contour Integration		418
XIII	Exercises		420
XIII	LACIOIGGS		420
Cau	chy's Formula and Its Applications		427
1	Cauchy's Formula		427
	1 Introduction2 First Explanation		427 427
	3 Gauss' Mean Value Theorem		429
	4 General Cauchy Formula		429

Infinite Differentiability and Taylor Series

Infinite Differentiability
Taylor Series

Calculus of Residues

431 431

432

434

9

oc II

S 111

2

xxii Contents

		2 A Formula3 Application4 Calculating	eries Centred at a Pole for Calculating Residues to Real Integrals Residues using Taylor Series to Summation of Series	434 435 436 438 439
	IV	Annular Laurei 1 An Exampl 2 Laurent's T	e	442 442 442
	V	Exercises		446
10	Vec	or Fields: Ph	ysics and Topology	450
	1	2 Physical Ve	functions as Vector Fields ector Fields Force Fields nd Sinks	450 450 451 453 454
	II	1 The Index	pers and Vector Fields* of a Singular Point According to Poincaré Theorem	456 456 459 460
	Ш	2 Defining th	ed Surfaces* n of the Poincaré-Hopf Theorem e Index on a Surface ation of the Poincaré-Hopf Theorem	462 462 464 465
	IV	Exercises		468
11	Vec	or Fields and	Complex Integration	472
	I	4 Divergence	and Local Work e and Curl in Geometric Form* e-Free and Curl-Free Vector Fields	472 472 474 476 478 479
	H		ration in Terms of Vector Fields Vector Field	481 481
	III	4 Example: \(\) 5 Local Beha 6 Cauchy's F 7 Positive Po 8 Negative P 9 Multipoles	Area as Flux Winding Number as Flux aviour of Vector Fields* Formula owers Powers and Multipoles at Infinity Geries as a Multipole Expansion	483 484 485 486 488 489 490 492 493

				Contents	xxiii
		2 The 3 The 4 The 5 The	oduction Stream Function Gradient Field Potential Function Complex Potential		494 494 497 498 500 503
	IV	Exercise	S		505
12	Flov	vs and H	armonic Functions		508
	I		c Duals al Flows monic Duals		508 508 511
	H	1 Cor 2 Cor	al Invariance nformal Invariance of Harmonicity nformal Invariance of the Laplacian Meaning of the Laplacian		513 513 515 516
	111	A Power	ful Computational Tool		517
	IV	1 Sor 2 The 3 Fur	nplex Curvature Revisited* me Geometry of Harmonic Equipotentials c Curvature of Harmonic Equipotentials ther Complex Curvature Calculations ther Geometry of the Complex Curvature		520 520 520 523 525
	V	1 Intr 2 An 3 The	ound an Obstacle oduction Example Method of Images pping One Flow Onto Another		527 527 527 532 538
	VI	1 Intr 2 Ext 3 Inte 4 Inte 5 An	sics of Riemann's Mapping Theoren oduction erior Mappings and Flows Round Obstacles erior Mappings and Dipoles erior Mappings, Vortices, and Sources Example: Automorphisms of the Disc een's Function	n	540 540 541 544 546 549 550
	VII	1 Intr 2 Sch 3 Diri 4 The	s Problem oduction nwarz's Interpretation chlet's Problem for the Disc e Interpretations of Neumann and Bôcher een's General Formula		554 554 556 558 560 565
	VIII	Exercise	S		570
Re	fere	nces			573
Inc	lex				579