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¹Mathematics in the 20th Century (Shanitzer & Stillman, 2000, p. 6)

²The full quotation begins to reveal Cartan's heroism: "The utility of the absolute differential calculus of Ricci and Levi-Civita must be tempered by an avoidance of excessive formal calculations, where the debauch of indices disguises an often very simple geometric reality. It is this reality that I have sought to reveal." (From the preface to Cartan 1928.)

³Given the frequency with which I shall have occasion to refer to it in my first book, *Visual Complex Analysis* (Needham 1997),

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