

# FIXED POINT THEORY AND GRAPH THEORY FOUNDATIONS AND INTEGRATIVE APPROACHES

Edited by **Monther Rashed Alfuraidan and Qamrul Hasan Ansari**

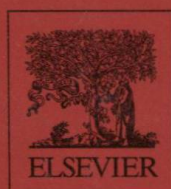
***Fixed Point Theory and Graph Theory: Foundations and Integrative Approaches*** links the theories of fixed point theorems, which give the conditions under which maps (single or multivalued) have solutions, and graph theory, which uses mathematical structures to illustrate the relationship between ordered pairs of objects in terms of their vertices and directed edges.

This edited reference work is perhaps the first to provide a link between the two theories, describing not only their foundational aspects, but also the most recent advances and the fascinating intersection of the domains.

The authors provide solution methods for fixed points in different settings, with two chapters devoted to the solutions method for critically important non-linear problems in engineering, including, variational inequalities, fixed point, split feasibility, and hierarchical variational inequality problems. The last two chapters are devoted to integrating fixed point theory in spaces with the graph and the use of retractions in the fixed point theory for ordered sets.

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