FIXED POINT THEORY AND GRAPH THEORY

FOUNDATIONS AND INTEGRATIVE APPROACHES

Edited by **Monther Rashed Alfuraidan and Qamrul Hasan Ansari**

Fixed Point Theory and Graph Theory: Foundations and Integrative Approaches links the theories of fixed point theorems, which give the conditions under which maps (single or multivalued) have solutions, and graph theory, which uses mathematical structures to illustrate the relationship between ordered pairs of objects in terms of their vertices and directed edges.

This edited reference work is perhaps the first to provide a link between the two theories, describing not only their foundational aspects, but also the most recent advances and the fascinating intersection of the domains.

The authors provide solution methods for fixed points in different settings, with two chapters devoted to the solutions method for critically important non-linear problems in engineering, including, variational inequalities, fixed point, split feasibility, and hierarchical variational inequality problems. The last two chapters are devoted to integrating fixed point theory in spaces with the graph and the use of retractions in the fixed point theory for ordered sets.

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CONTENTS

Contents v
Foreword ix
Acknowledgments xi
Preface xiii
About the Authors xv

1. Caristi-Browder Operator Theory in Distance Spaces 1
Vasile Berinde, and Ioan A. Rus

1.1. From the Caristi Fixed Point Theorems to Caristi, Caristi-Kirk and Caristi-Browder Operators 1
1.2. List of Notations 4
1.3. Weakly Picard Operators on L-Spaces 4
1.4. Caristi-Browder Operators on Metric Spaces 6
1.5. Caristi-Browder Operators on $\mathbb{R}_+^m$-Metric Spaces 12
1.6. Caristi-Browder Operators on $s(\mathbb{R}_+)$-Metric Spaces 16
1.7. Caristi-Browder Operators on Kasahara Spaces 19
1.8. Research Directions in the Caristi-Browder Operator Theory 21
References 22

2. Iterative Approximation of Fixed Points of Single-valued Almost Contractions 29
Vasile Berinde, and Mădălina Păcurar

2.1. Introduction 29
2.2. Fixed Point Theorems for Single-valued Self Almost Contractions 32
2.3. Implicit Almost Contractions 50
2.4. Common Fixed Point Theorems for Almost Contractions 58
2.5. Almost Contractive type Mappings on Product Spaces 65
2.6. Fixed Point Theorems for Single-valued Nonself Almost Contractions 81
References 87

3. Approximate Fixed Points 99
Mostafa Bachar, Buthinah A. Bin Dehaish and Mohamed Amine Khamsi

3.1. Introduction 99
3.2. Approximate Fixed Points of Mappings in Banach Spaces 101
3.3. Approximate Fixed Points of Mappings in Hyperbolic Spaces 107
3.4. Approximate Fixed Points of Monotone Mappings 117

3.5. Approximate Fixed Points of Nonlinear Semigroups 124
3.6. Approximate Fixed Points of Monotone Nonlinear Semigroups 129
References 136

4. Viscosity Methods for Some Applied Nonlinear Analysis Problems 139
Paul-Emile Maingé and Abdellatif Moudafi

4.1. Introduction 140
4.2. Viscosity Method for Fixed Point Problems 142
4.3. Viscosity Method for Split Common Fixed Point Problems 154
4.3.1. Algorithms and Related Convergence Results 154
4.3.2. Convergence Results in Some Specific Cases 163
4.3.3. Numerical Experiments for an Inverse Heat Equation 165
4.4. Viscosity Method for Split Equilibrium Problems 170
References 182

5. Extragradient Methods for Some Nonlinear Problems 187
Qamrul Hasan Ansari and D. R. Sahu

5.1. Introduction 188
5.2. Preliminaries 189
5.3. Projection Gradient Method 194
5.4. Extragradient Method for Nonexpansive Mappings and Variational Inequalities 196
5.5. Modified Extragradient Method for Nonexpansive Mappings and Variational Inequalities 201
5.6. Extragradient Method for Hierarchical Variational Inequalities 206
5.6.1. Implicit Extragradient Method for Hierarchical Variational Inequalities 207
5.6.2. Explicit Extragradient Method for Hierarchical Variational Inequalities 213
5.7. Extragradient Methods for Split Feasibility Problems 218
5.7.1. An Extragradient Method for Nonexpansive Mappings and Split Feasibility Problems 220
5.7.2. Strong Convergence of a Relaxed Extragradient Method for Split Feasibility Problems 225
References 228

6. Iterative Methods for Nonexpansive Type Mappings 231
Abdul Rahim Khan and Hafiz Fukhar-ud-din

6.1. Introduction and Preliminaries 231
6.2. Nonexpansive Mappings in Uniformly Convex Banach Spaces 235
6.3. Nonexpansive Mappings in CAT(0) Spaces 244
6.4. An Algorithm of Asymptotically Nonexpansive Mappings 253

6.5. Existence and Approximation of Fixed Points 265
6.6. Viscosity Method for Generalized Asymptotically Nonexpansive Mappings 272
References 283

7. Metric Fixed Point Theory in Spaces with a Graph 287
Monther Rashed Alfuraidan

7.1. Introduction 288
7.2. Banach Contraction Principle 290
7.2.1. Ran and Reurings Fixed Point Theorem 291
7.2.2. Nieto and Rodríguez-López Fixed Point Theorem 292
7.3. Basic Definitions and Properties 294
7.3.1. Graph Theory 294
7.3.2. Hyperbolic Metric Spaces 296
7.3.3. Modular Function Spaces 297
7.3.4. Modular Metric Spaces 301
7.4. Banach Contraction Principle in Metric Spaces with a Graph 305
7.5. Caristi's Fixed Point Theorem 305
7.5.1. Caristi's Theorem in Partially Ordered Metric Spaces 306
7.5.2. Caristi's Theorem in Metric Spaces with a Graph 310
7.5.3. Minimal Points and Fixed Point Property in a Graph 311
7.5.4. Kirk's Problem via Graphs 314
7.5.5. Applications of Metric Spaces with a Graph 316
7.6. The Contraction Principle in Modular Metric Spaces with a Graph 319
7.7. Monotone Pointwise Contractions in Banach Spaces with a Graph 323
7.8. Monotone Ćirić Quasi-Contraction Mappings 331
7.8.1. Monotone Quasi-Contraction Mappings in Metric Spaces with a Graph 331
7.8.2. Monotone Quasi-Contraction Mappings in Modular Metric Spaces with a Graph 334
7.9. Monotone Nonexpansive Mappings in Banach Spaces with a Graph 337
7.10. Monotone Nonexpansive Mappings in Hyperbolic Metric Spaces with a Graph 340
7.11. The Contraction Principle for Monotone Multivalued Mappings 343
7.11.1. Monotone Multivalued Mappings in Metric Spaces with a Graph 343
7.11.2. The Contraction Principle for Multivalued Mappings in Modular Function Spaces with a Graph 347
7.11.3. The Contraction Principle for Multivalued Mappings in Modular Metric Spaces with a Graph 352
7.12. Monotone Nonexpansive Multivalued Mappings 355
7.12.1. Monotone Nonexpansive Multivalued Mappings in Hyperbolic Metric Spaces with a Graph 355
7.12.2. Monotone Nonexpansive Multivalued Mappings in Modular Function Spaces with a Graph 357
References 361

8. The Use of Retractions in the Fixed Point Theory for Ordered Sets 365
Bernd S. W. Schröder

8.1. Introduction 366
8.2. Chain-Complete Ordered Sets 368
8.3. The Abian-Brown Theorem 370
8.4. Comparative Retractions 373
8.5. Irreducible Points 376
8.6. Constraint Propagation 379
8.7. Retractable Points 383
8.8. Verifying Connected Collapsibility Directly 387
8.9. Graphs 392
8.10. Simplicial Complexes 399
8.11. Topological Realizations 405
8.12. Iterated Clique Graphs 406
8.13. K-Null Comparability Graphs 414
References 416

Index 419