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Lessons in Enumerative Combinatorics

This textbook introduces enumerative combinatorics through the framework of formal languages and bijections. By starting with elementary operations on words and languages, the authors paint an insightful, unified picture for readers entering the field. Numerous concrete examples and illustrative metaphors motivate the theory throughout, while the overall approach illuminates the important connections between discrete mathematics and theoretical computer science.

Beginning with the basics of formal languages, the first chapter quickly establishes a common setting for modeling and counting classical combinatorial objects and constructing bijective proofs. From here, topics are modular and offer substantial flexibility when designing a course. Chapters on generating functions and partitions build further fundamental tools for enumeration and include applications such as a combinatorial proof of the Lagrange inversion formula. Connections to linear algebra emerge in chapters studying Cayley trees, determinantal formulas, and the combinatorics that lie behind the classical Cayley–Hamilton theorem. The remaining chapters range across the Inclusion-Exclusion Principle, graph theory and coloring, exponential structures, matching and distinct representatives, with each topic opening many doors to further study. Generous exercise sets complement all chapters, and miscellaneous sections explore additional applications.

Lessons in Enumerative Combinatorics captures the authors' distinctive style and flair for introducing newcomers to combinatorics. The conversational yet rigorous presentation suits students in mathematics and computer science at the graduate, or advanced undergraduate level. Knowledge of single-variable calculus and the basics of discrete mathematics is assumed; familiarity with linear algebra will enhance the study of certain chapters.

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