

Contents

Part I. Introduction

Prologue	1
1. Historical Introduction	21
1.1 The Revolution of Physics	22
1.2 Quantization in a Nutshell	27
1.2.1 Basic Formulas.....	30
1.2.2 The Fundamental Role of the Harmonic Oscillator in Quantum Field Theory	46
1.2.3 Quantum Fields and Second Quantization	52
1.2.4 The Importance of Functional Integrals	57
1.3 The Role of Göttingen	60
1.4 The Göttingen Tragedy	67
1.5 Highlights in the Sciences.....	69
1.5.1 The Nobel Prize.....	69
1.5.2 The Fields Medal in Mathematics	71
1.5.3 The Nevanlinna Prize in Computer Sciences	72
1.5.4 The Wolf Prize in Physics	73
1.5.5 The Wolf Prize in Mathematics	73
1.5.6 The Abel Prize in Mathematics	75
1.6 The Emergence of Physical Mathematics – a New Dimension of Mathematics	75
1.7 The Seven Millennium Prize Problems of the Clay Mathematics Institute.....	77
2. Phenomenology of the Standard Model for Elementary Particles	79
2.1 The System of Units	80
2.2 Waves in Physics	81
2.2.1 Harmonic Waves	81
2.2.2 Wave Packets	82
2.2.3 Standing Waves	84
2.2.4 Electromagnetic Waves.....	85
2.2.5 Superposition of Waves and the Fourier Transform ...	86

2.2.6	Damped Waves, the Laplace Transform, and Dispersion Relations	89
2.2.7	The Response Function, the Feynman Propagator, and Causality	94
2.3	Historical Background.....	97
2.3.1	Planck's Radiation Law	101
2.3.2	The Boltzmann Statistics and Planck's Quantum Hypothesis	106
2.3.3	Einstein's Theory of Special Relativity	109
2.3.4	Einstein's Theory of General Relativity	111
2.3.5	Einstein's Light Particle Hypothesis	112
2.3.6	Rutherford's Particle Scattering	113
2.3.7	The Cross Section for Compton Scattering.....	115
2.3.8	Bohr's Model of the Hydrogen Atom	120
2.3.9	Einstein's Radiation Law and Laser Beams	124
2.3.10	Quantum Computers	126
2.4	The Standard Model in Particle Physics	127
2.4.1	The Four Fundamental Forces in Nature	127
2.4.2	The Fundamental Particles in Nature	130
2.5	Magic Formulas	140
2.6	Quantum Numbers of Elementary Particles	143
2.6.1	The Spin	144
2.6.2	Conservation of Quantum Numbers	154
2.7	The Fundamental Role of Symmetry in Physics	162
2.7.1	Classical Symmetries	168
2.7.2	The CPT Symmetry Principle for Elementary Particles	170
2.7.3	Local Gauge Symmetry	174
2.7.4	Permutations and Pauli's Exclusion Principle	176
2.7.5	Crossing Symmetry	176
2.7.6	Forbidden Spectral Lines in Molecules	177
2.8	Symmetry Breaking.....	178
2.8.1	Parity Violation and CP Violation	178
2.8.2	Irreversibility	179
2.8.3	Splitting of Spectral Lines in Molecules	179
2.8.4	Spontaneous Symmetry Breaking and Particles	180
2.8.5	Bifurcation and Phase Transitions	182
2.9	The Structure of Interactions in Nature	183
2.9.1	The Electromagnetic Field as Generalized Curvature ..	183
2.9.2	Physics and Modern Differential Geometry	184
3.	The Challenge of Different Scales in Nature	187
3.1	The Trouble with Scale Changes	187
3.2	Wilson's Renormalization Group Theory in Physics	189
3.2.1	A New Paradigm in Physics	191

3.2.2	Screening of the Coulomb Field and the Renormalization Group of Lie Type	193
3.2.3	The Running Coupling Constant and the Asymptotic Freedom of Quarks	201
3.2.4	The Quark Confinement	204
3.2.5	Proton Decay and Supersymmetric Grand Unification ..	205
3.2.6	The Adler–Bell–Jackiw Anomaly	205
3.3	Stable and Unstable Manifolds	206
3.4	A Glance at Conformal Field Theories	207

Part II. Basic Techniques in Mathematics

4.	Analyticity	209
4.1	Power Series Expansion	210
4.2	Deformation Invariance of Integrals	212
4.3	Cauchy’s Integral Formula	212
4.4	Cauchy’s Residue Formula and Topological Charges	213
4.5	The Winding Number	214
4.6	Gauss’ Fundamental Theorem of Algebra	215
4.7	Compactification of the Complex Plane	217
4.8	Analytic Continuation and the Local-Global Principle	218
4.9	Integrals and Riemann Surfaces	219
4.10	Domains of Holomorphy	223
4.11	A Glance at Analytic <i>S</i> -Matrix Theory	224
4.12	Important Applications	225
5.	A Glance at Topology	227
5.1	Local and Global Properties of the Universe	227
5.2	Bolzano’s Existence Principle	228
5.3	Elementary Geometric Notions	230
5.4	Manifolds and Diffeomorphisms	234
5.5	Topological Spaces, Homeomorphisms, and Deformations ..	235
5.6	Topological Quantum Numbers	241
5.6.1	The Genus of a Surface	241
5.6.2	The Euler Characteristic	242
5.6.3	Platonic Solids and Fullerenes	244
5.6.4	The Poincaré–Hopf Theorem for Velocity Fields	245
5.6.5	The Gauss–Bonnet Theorem	246
5.6.6	The Morse Theorem on Critical Points of Energy Functions	250
5.6.7	Magnetic Fields, the Gauss Integral, and the Linking Number	251
5.6.8	Electric Fields, the Kronecker Integral, and the Mapping Degree	253

5.6.9	The Heat Kernel and the Atiyah–Singer Index Theorem	257
5.6.10	Knots and Topological Quantum Field Theory	263
5.7	Quantum States	265
5.7.1	The Topological Character of the Electron Spin	265
5.7.2	The Hopf Fibration of the 3-Dimensional Sphere	268
5.7.3	The Homotopy Functor	271
5.7.4	Grassmann Manifolds and Projective Geometry	274
5.8	Perspectives	275
6.	Many-Particle Systems in Mathematics and Physics	277
6.1	Partition Function in Statistical Physics	279
6.2	Euler’s Partition Function	283
6.3	Discrete Laplace Transformation	285
6.4	Integral Transformations	289
6.5	The Riemann Zeta Function	291
6.5.1	The Prime Number Theorem – a Pearl of Mathematics	291
6.5.2	The Riemann Hypothesis	296
6.5.3	Dirichlet’s L -Function	296
6.6	The Casimir Effect in Quantum Field Theory and the Epstein Zeta Function	299
6.7	Appendix: The Mellin Transformation and Other Useful Analytic Techniques by Don Zagier	305
6.7.1	The Generalized Mellin Transformation	305
6.7.2	Dirichlet Series and their Special Values	309
6.7.3	Application: the Casimir Effect	312
6.7.4	Asymptotics of Series of the Form $\sum f(nt)$	317
7.	Rigorous Finite-Dimensional Magic Formulas of Quantum Field Theory	325
7.1	Geometrization of Physics	325
7.2	Ariadne’s Thread in Quantum Field Theory	326
7.3	Linear Spaces	328
7.4	Finite-Dimensional Hilbert Spaces	335
7.5	Groups	340
7.6	Lie Algebras	342
7.7	Lie’s Logarithmic Trick for Matrix Groups	345
7.8	Lie Groups	347
7.9	Basic Notions in Quantum Physics	349
7.9.1	States, Costates, and Observables	350
7.9.2	Observers and Coordinates	354
7.10	Fourier Series	355
7.11	Dirac Calculus in Finite-Dimensional Hilbert Spaces	359
7.12	The Trace of a Linear Operator	363
7.13	Banach Spaces	366

7.14	Probability and Hilbert's Spectral Family of an Observable	368
7.15	Transition Probabilities, <i>S</i> -Matrix, and Unitary Operators	370
7.16	The Magic Formulas for the Green's Operator	372
7.16.1	Non-Resonance and Resonance	373
7.16.2	Causality and the Laplace Transform	377
7.17	The Magic Dyson Formula for the Retarded Propagator	381
7.17.1	Lagrange's Variation of the Parameter	383
7.17.2	Duhamel's Principle	385
7.17.3	The Volterra Integral Equation	386
7.17.4	The Dyson Series	388
7.18	The Magic Dyson Formula for the <i>S</i> -Matrix	390
7.19	Canonical Transformations	392
7.19.1	The Schrödinger Picture	392
7.19.2	The Heisenberg Picture	393
7.19.3	The Dirac Interaction Picture	394
7.20	Functional Calculus	395
7.20.1	Functional Derivatives	396
7.20.2	Partial Functional Derivatives	401
7.20.3	Infinitesimal Transformations	409
7.20.4	Functional Integration	416
7.21	The Discrete Feynman Path Integral	416
7.21.1	The Magic Feynman Propagator Formula	417
7.21.2	The Magic Formula for Time-Ordered Products	422
7.21.3	The Trace Formula	423
7.22	Causal Correlation Functions	424
7.22.1	The Wick Rotation Trick for Vacuum Expectation Values	425
7.22.2	The Magic Gell-Mann–Low Reduction Formula	427
7.23	The Magic Gaussian Integral	428
7.23.1	The One-Dimensional Prototype	428
7.23.2	The Determinant Trick	434
7.23.3	The Zeta Function Trick	434
7.23.4	The Moment Trick	435
7.23.5	The Method of Stationary Phase	435
7.24	The Rigorous Response Approach to Finite Quantum Fields	438
7.24.1	Basic Ideas	439
7.24.2	Discrete Space-Time Manifold	441
7.24.3	The Principle of Critical Action	445
7.24.4	The Response Function	446
7.24.5	The Global Quantum Action Principle	447
7.24.6	The Magic Quantum Action Reduction Formula for Correlation Functions	448
7.24.7	The Magic LSZ Reduction Formula for Scattering Functions	449

7.24.8	The Local Quantum Action Principle	452
7.24.9	Simplifying the Computation of Quantum Effects	454
7.24.10	Reduced Correlation Functions	455
7.24.11	The Mean Field Approximation	456
7.24.12	Vertex Functions and the Effective Action	457
7.25	The Discrete φ^4 -Model and Feynman Diagrams	459
7.26	The Extended Response Approach	477
7.27	Complex-Valued Fields	483
7.28	The Method of Lagrange Multipliers	487
7.29	The Formal Continuum Limit	492
8.	Rigorous Finite-Dimensional Perturbation Theory	497
8.1	Renormalization	497
8.1.1	Non-Resonance	497
8.1.2	Resonance, Regularizing Term, and Bifurcation	499
8.1.3	The Renormalization Group	502
8.1.4	The Main Bifurcation Theorem	503
8.2	The Rellich Theorem	506
8.3	The Trotter Product Formula	507
8.4	The Magic Baker–Campbell–Hausdorff Formula	508
8.5	Regularizing Terms	509
8.5.1	The Weierstrass Product Theorem	509
8.5.2	The Mittag–Leffler Theorem	510
8.5.3	Regularization of Divergent Integrals	511
8.5.4	The Polchinski Equation	513
9.	Fermions and the Calculus for Grassmann Variables	515
9.1	The Grassmann Product	515
9.2	Differential Forms	516
9.3	Calculus for One Grassmann Variable	516
9.4	Calculus for Several Grassmann Variables	517
9.5	The Determinant Trick	518
9.6	The Method of Stationary Phase	519
9.7	The Fermionic Response Model	519
10.	Infinite-Dimensional Hilbert Spaces	521
10.1	The Importance of Infinite Dimensions in Quantum Physics	521
10.1.1	The Uncertainty Relation	521
10.1.2	The Trouble with the Continuous Spectrum	524
10.2	The Hilbert Space $L_2(\Omega)$	525
10.2.1	Measure and Integral	527
10.2.2	Dirac Measure and Dirac Integral	529
10.2.3	Lebesgue Measure and Lebesgue Integral	530
10.2.4	The Fischer–Riesz Theorem	531
10.3	Harmonic Analysis	532

10.3.1	Gauss' Method of Least Squares	532
10.3.2	Discrete Fourier Transform	533
10.3.3	Continuous Fourier Transform	535
10.4	The Dirichlet Problem in Electrostatics as a Paradigm	540
10.4.1	The Variational Lemma	542
10.4.2	Integration by Parts	544
10.4.3	The Variational Problem	547
10.4.4	Weierstrass' Counterexample	549
10.4.5	Typical Difficulties	552
10.4.6	The Functional Analytic Existence Theorem	555
10.4.7	Regularity of the Solution	558
10.4.8	The Beauty of the Green's Function	560
10.4.9	The Method of Orthogonal Projection	564
10.4.10	The Power of Ideas in Mathematics	567
10.4.11	The Ritz Method	568
10.4.12	The Main Existence Principle	569
11.	Distributions and Green's Functions	575
11.1	Rigorous Basic Ideas	579
11.1.1	The Discrete Dirac Delta Function	580
11.1.2	Prototypes of Green's Functions	581
11.1.3	The Heat Equation and the Heat Kernel	586
11.1.4	The Diffusion Equation	587
11.1.5	The Schrödinger Equation and the Euclidean Approach	588
11.2	Dirac's Formal Approach	589
11.2.1	Dirac's Delta Function	590
11.2.2	Density of a Mass Distribution	591
11.2.3	Local Functional Derivative	591
11.2.4	The Substitution Rule	595
11.2.5	Formal Dirac Calculus and the Fourier Transform	596
11.2.6	Formal Construction of the Heat Kernel	606
11.3	Laurent Schwartz's Rigorous Approach	607
11.3.1	Physical Measurements and the Idea of Averaging	607
11.3.2	Distributions	608
11.3.3	Tempered Distributions	614
11.3.4	The Fourier Transform	617
11.4	Hadamard's Regularization of Integrals	618
11.4.1	Regularization of Divergent Integrals	618
11.4.2	The Sokhotski Formula	619
11.4.3	Steinmann's Renormalization Theorem	620
11.4.4	Regularization Terms	622
11.5	Renormalization of the Anharmonic Oscillator	625
11.5.1	Renormalization in a Nutshell	625
11.5.2	The Linearized Problem	625

11.5.3	The Nonlinear Problem and Non-Resonance	629
11.5.4	The Nonlinear Problem, Resonance, and Bifurcation ..	630
11.5.5	The Importance of the Renormalized Green's Function	632
11.5.6	The Renormalization Group	633
11.6	The Importance of Algebraic Feynman Integrals	634
11.6.1	Wick Rotation and Cut-Off	634
11.6.2	Dimensional Regularization	636
11.6.3	Weinberg's Power-Counting Theorem	638
11.6.4	Integration Tricks	640
11.7	Fundamental Solutions of Differential Equations	644
11.7.1	The Newtonian Potential	646
11.7.2	The Existence Theorem	646
11.7.3	The Beauty of Hironaka's Theorem	647
11.8	Functional Integrals	651
11.8.1	The Feynman Path Integral for the Heat Equation	651
11.8.2	Diffusion, Brownian Motion, and the Wiener Integral .	654
11.8.3	The Method of Quantum Fluctuations	655
11.8.4	Infinite-Dimensional Gaussian Integrals and Zeta Function Regularization	657
11.8.5	The Euclidean Trick and the Feynman Path Integral for the Schrödinger Equation	658
11.9	A Glance at Harmonic Analysis	660
11.9.1	The Fourier–Laplace Transform	660
11.9.2	The Riemann–Hilbert Problem	662
11.9.3	The Hilbert Transform	663
11.9.4	Symmetry and Special Functions	664
11.9.5	Tempered Distributions as Boundary Values of Ana- lytic Functions	665
11.10	The Trouble with the Euclidean Trick	666
12.	Distributions and Physics	669
12.1	The Discrete Dirac Calculus	669
12.1.1	Lattices	669
12.1.2	The Four-Dimensional Discrete Dirac Delta Function .	670
12.1.3	Rigorous Discrete Dirac Calculus	673
12.1.4	The Formal Continuum Limit	673
12.2	Rigorous General Dirac Calculus	675
12.2.1	Eigendistributions	675
12.2.2	Self-Adjoint Operators	677
12.2.3	The von Neumann Spectral Theorem	678
12.2.4	The Gelfand–Kostyuchenko Spectral Theorem	679
12.2.5	The Duality Map	679
12.2.6	Dirac's Notation	680
12.2.7	The Schwartz Kernel Theorem	681
12.3	Fundamental Limits in Physics	682

12.3.1	High-Energy Limit	682
12.3.2	Thermodynamic Limit and Phase Transitions	682
12.3.3	Adiabatic Limit	685
12.3.4	Singular Limit	689
12.4	Duality in Physics	690
12.4.1	Particles and de Broglie's Matter Waves	690
12.4.2	Time and Frequency	692
12.4.3	Time and Energy	692
12.4.4	Position and Momentum	692
12.4.5	Causality and Analyticity	695
12.4.6	Strong and Weak Interaction	702
12.5	Microlocal Analysis	703
12.5.1	Singular Support of a Distribution	704
12.5.2	Wave Front Set	706
12.5.3	The Method of Stationary Phase	714
12.5.4	Short-Wave Asymptotics for Electromagnetic Waves	718
12.5.5	Diffraction of Light	724
12.5.6	Pseudo-Differential Operators	728
12.5.7	Fourier Integral Operators	728
12.6	Multiplication of Distributions	729
12.6.1	Laurent Schwartz's Counterexample	730
12.6.2	Hörmander's Causal Product	732

Part III. Heuristic Magic Formulas of Quantum Field Theory

13.	Basic Strategies in Quantum Field Theory	739
13.1	The Method of Moments and Correlation Functions	742
13.2	The Power of the <i>S</i> -Matrix	745
13.3	The Relation Between the <i>S</i> -Matrix and the Correlation Functions	746
13.4	Perturbation Theory and Feynman Diagrams	747
13.5	The Trouble with Interacting Quantum Fields	748
13.6	External Sources and the Generating Functional	749
13.7	The Beauty of Functional Integrals	751
13.7.1	The Principle of Critical Action	752
13.7.2	The Magic Feynman Representation Formula	753
13.7.3	Perturbation Theory	754
13.7.4	Renormalization	754
13.7.5	Transition Amplitudes	755
13.7.6	The Magic Trace Formula	756
13.8	Quantum Field Theory at Finite Temperature	757
13.8.1	The Partition Function	757
13.8.2	The Classical Hamiltonian Approach	760

13.8.3	The Magic Feynman Functional Integral for the Partition Function	761
13.8.4	The Thermodynamic Limit	763
14.	The Response Approach	765
14.1	The Fourier–Minkowski Transform	770
14.2	The φ^4 -Model	773
14.2.1	The Classical Principle of Critical Action	774
14.2.2	The Response Function and the Feynman Propagator ..	774
14.2.3	The Extended Quantum Action Functional	782
14.2.4	The Magic Quantum Action Reduction Formula for Correlation Functions	782
14.2.5	The Magic LSZ Reduction Formula for the <i>S</i> -Matrix ..	785
14.2.6	The Local Quantum Action Principle	787
14.2.7	The Mnemonic Functional Integral	787
14.2.8	Bose–Einstein Condensation of Dilute Gases	788
14.3	A Glance at Quantum Electrodynamics	789
14.3.1	The Equations of Motion	791
14.3.2	The Principle of Critical Action	792
14.3.3	The Gauge Field Approach	794
14.3.4	The Extended Action Functional with Source Term ..	797
14.3.5	The Response Function for Photons	799
14.3.6	The Response Function for Electrons	800
14.3.7	The Extended Quantum Action Functional	801
14.3.8	The Magic Quantum Action Reduction Formula	803
14.3.9	The Magic LSZ Reduction Formula	803
14.3.10	The Mnemonic Functional Integral	804
15.	The Operator Approach	813
15.1	The φ^4 -Model	814
15.1.1	The Lattice Approximation	815
15.1.2	Fourier Quantization	817
15.1.3	The Free 2-Point Green’s Function	820
15.1.4	The Magic Dyson Formula for the <i>S</i> -Matrix	822
15.1.5	The Main Wick Theorem	824
15.1.6	Transition Amplitude	829
15.1.7	Transition Probability	837
15.1.8	Scattering Cross Section	839
15.1.9	General Feynman Rules for Particle Scattering	843
15.1.10	The Magic Gell-Mann–Low Reduction Formula for Green’s Functions	845
15.2	A Glance at Quantum Electrodynamics	846
15.3	The Role of Effective Quantities in Physics	847
15.4	A Glance at Renormalization	848
15.4.1	The Trouble with the Continuum Limit	850

15.4.2 Basic Ideas of Renormalization	850
15.4.3 The BPHZ Renormalization	853
15.4.4 The Epstein–Glaser Approach	854
15.4.5 Algebraic Renormalization	858
15.4.6 The Importance of Hopf Algebras	859
15.5 The Convergence Problem in Quantum Field Theory	860
15.5.1 Dyson’s No-Go Argument	860
15.5.2 The Power of the Classical Ritt Theorem in Quantum Field Theory	861
15.6 Rigorous Perspectives	862
15.6.1 Axiomatic Quantum Field Theory	866
15.6.2 The Euclidean Strategy in Constructive Quantum Field Theory	870
15.6.3 The Renormalization Group Method	872
16. Peculiarities of Gauge Theories	877
16.1 Basic Difficulties	877
16.2 The Principle of Critical Action	878
16.3 The Language of Physicists	884
16.4 The Importance of the Higgs Particle	886
16.5 Integration over Orbit Spaces	886
16.6 The Magic Faddeev–Popov Formula and Ghosts	888
16.7 The BRST Symmetry	890
16.8 The Power of Cohomology	891
16.8.1 Physical States, Unphysical States, and Cohomology	893
16.8.2 Forces and Potentials	894
16.8.3 The Cohomology of Geometric Objects	896
16.8.4 The Spectra of Atoms and Cohomology	899
16.8.5 BRST Symmetry and the Cohomology of Lie Groups	900
16.9 The Batalin–Vilkovisky Formalism	903
16.10 A Glance at Quantum Symmetries	904
17. A Panorama of the Literature	907
17.1 Introduction to Quantum Field Theory	907
17.2 Standard Literature in Quantum Field Theory	910
17.3 Rigorous Approaches to Quantum Field Theory	911
17.4 The Fascinating Interplay between Modern Physics and Mathematics	913
17.5 The Monster Group, Vertex Algebras, and Physics	919
17.6 Historical Development of Quantum Field Theory	924
17.7 General Literature in Mathematics and Physics	925
17.8 Encyclopedias	926
17.9 Highlights of Physics in the 20th Century	926
17.10 Actual Information	928

Appendix	931
A.1 Notation	931
A.2 The International System of Units	934
A.3 The Planck System	936
A.4 The Energetic System	942
A.5 The Beauty of Dimensional Analysis	944
A.6 The Similarity Principle in Physics	946
Epilogue	955
References	959
List of Symbols	991
Index	995