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Is Euclidean geometry true? It has no meaning. . . . One geometry cannot be more true than another; it can only be more convenient.
—H. Poincaré, 1902

I attach special importance to the view of geometry which I have just set forth, because without it I should have been unable to formulate the theory of relativity.
—A. Einstein, 1922

Although several mathematicians, especially O.F. Gauss, studied the notion of curvature in special cases, the idea of an abstract manifold of arbitrary finite dimension seems to be due to Riemann. Mathematicians were led to these notions only slowly. As the idea of a vector space of dimension higher than three became acceptable in the last century, algebraic geometers began to study the solutions of polynomial equations in many variables. For example, they studied the algebraic curves in the complex projective plane, which in the real sense is roughly the study of ordinary surfaces in four-dimensional space. At the same time, mathematical physicists interested themselves in six-dimensional space, the state space of a single particle with three position and three momentum variables; if