

# Contents

|                                                                          |           |
|--------------------------------------------------------------------------|-----------|
| <b>Preface</b>                                                           | <b>v</b>  |
| <b>1 Introduction</b>                                                    | <b>1</b>  |
| <b>2 Least squares methods</b>                                           | <b>3</b>  |
| 2.1 History of least squares methods                                     | 4         |
| 2.1.1 Classical least squares method                                     | 4         |
| 2.1.2 Total least squares method                                         | 5         |
| 2.2 Total least squares - the method                                     | 6         |
| 2.2.1 Singular value decomposition                                       | 6         |
| 2.2.2 The line of total least squares method                             | 7         |
| 2.3 Comparison of least squares methods                                  | 9         |
| 2.3.1 Simple regression in total least squares sense                     | 11        |
| 2.3.2 The method of least circles                                        | 11        |
| 2.3.3 Numerical point of view                                            | 12        |
| 2.3.4 The weighed form of least squares methods                          | 13        |
| <b>3 Fractional calculus</b>                                             | <b>15</b> |
| 3.1 Properties                                                           | 15        |
| 3.2 Special functions                                                    | 16        |
| 3.2.1 Gamma function                                                     | 16        |
| 3.2.2 Beta function                                                      | 17        |
| 3.2.3 Mittag-Leffler function                                            | 17        |
| 3.3 Historical development of fractional calculus                        | 18        |
| 3.3.1 Riemann-Liouville definition                                       | 19        |
| 3.3.2 Grünwald-Letnikov definition                                       | 19        |
| 3.3.3 Caputo definition                                                  | 20        |
| 3.4 Transform methods                                                    | 21        |
| 3.4.1 Laplace transform method                                           | 21        |
| 3.4.2 Fourier transform method                                           | 22        |
| <b>4 Numerical methods for solving fractional differential equations</b> | <b>23</b> |
| 4.1 Fractional backward difference methods                               | 23        |
| 4.1.1 Backward differences based on the Grünwald-Letnikov definition     | 24        |
| 4.1.2 Diethelm's fractional backward differences                         | 25        |
| 4.1.3 Lubich's fractional backward difference methods                    | 26        |
| 4.1.4 Podlubny's 'matrix approach'                                       | 28        |
| 4.2 Other numerical methods and schemes                                  | 30        |

|          |                                                                          |           |
|----------|--------------------------------------------------------------------------|-----------|
| 4.2.1    | Adams-Bashforth-Moulton method . . . . .                                 | 31        |
| 4.2.2    | Generalized Taylor series expansion . . . . .                            | 33        |
| 4.2.3    | Adomian's decomposition method . . . . .                                 | 34        |
| <b>5</b> | <b>Identification of fractional-order systems . . . . .</b>              | <b>39</b> |
| 5.1      | Model-structure definition . . . . .                                     | 40        |
| 5.1.1    | Linear model . . . . .                                                   | 41        |
| 5.1.2    | Nonlinear model with cross-products . . . . .                            | 41        |
| 5.1.3    | Nonlinear model with square-products . . . . .                           | 42        |
| 5.2      | Discretizing fractional order derivatives . . . . .                      | 42        |
| 5.3      | Solving systems of FDEs using the Grünwald-Letnikov definition . . . . . | 43        |
| 5.3.1    | Solving systems of linear FDEs . . . . .                                 | 43        |
| 5.3.2    | Solving systems of nonlinear FDEs with cross-products . . . . .          | 44        |
| 5.3.3    | Solving systems of nonlinear FDEs with square-products . . . . .         | 45        |
| 5.4      | Solving systems of PDEs using Prilubny's "matrix approach" . . . . .     | 47        |
| 5.4.1    | Solving systems of linear FDEs . . . . .                                 | 47        |
| 5.4.2    | Solving systems of nonlinear FDEs with cross-products . . . . .          | 48        |
| 5.5      | Formulation of the fitting criterion . . . . .                           | 49        |
| 5.5.1    | Least squares method . . . . .                                           | 50        |
| 5.5.2    | Least "circles" method . . . . .                                         | 51        |
| 5.5.3    | Orthogonal distance fitting . . . . .                                    | 52        |
| 5.5.4    | "Ranged" orthogonal distance fitting . . . . .                           | 52        |
| 5.5.5    | Chosen fitting criterion . . . . .                                       | 53        |
| 5.6      | Testing the method of data fitting . . . . .                             | 54        |
| 5.6.1    | Data fitting in 2D space . . . . .                                       | 57        |
| 5.6.2    | Data fitting in 3D space . . . . .                                       | 64        |
| 5.6.3    | Results of data fitting . . . . .                                        | 71        |
| <b>6</b> | <b>Conclusion . . . . .</b>                                              | <b>73</b> |
|          | <b>Bibliography . . . . .</b>                                            | <b>75</b> |