

Contents

Prologue	xvii
Acknowledgements	xxv

ACT I

The Nature of Space

1	Euclidean and Non-Euclidean Geometry	3
1.1	Euclidean and Hyperbolic Geometry	3
1.2	Spherical Geometry	6
1.3	The Angular Excess of a Spherical Triangle	8
1.4	Intrinsic and Extrinsic Geometry of Curved Surfaces	9
1.5	Constructing Geodesics via Their Straightness	11
1.6	The Nature of Space	14
2	Gaussian Curvature	17
2.1	Introduction	17
2.2	The Circumference and Area of a Circle	19
2.3	The Local Gauss–Bonnet Theorem	22
	Exercises for Prologue and Act I	24

ACT II

The Metric

4	Mapping Surfaces: The Metric	31
4.1	Introduction	31
4.2	The Projective Map of the Sphere	32
4.3	The Metric of a General Surface	34
4.4	The Metric Curvature Formula	37
4.5	Conformal Maps	38
4.6	Some Visual Complex Analysis	41
4.7	The Conformal Stereographic Map of the Sphere	44
4.8	Stereographic Formulas	47
4.9	Stereographic Preservation of Circles	49
5	The Pseudosphere and the Hyperbolic Plane	51
5.1	Beltrami’s Insight	51
5.2	The Tractrix and the Pseudosphere	52
5.3	A Conformal Map of the Pseudosphere	54
5.4	The Beltrami–Poincaré Half-Plane	56

5.5	Using Optics to Find the Geodesics	58
5.6	The Angle of Parallelism	60
5.7	The Beltrami–Poincaré Disc	62
6	Isometries and Complex Numbers	65
6.1	Introduction	65
6.2	Möbius Transformations	67
6.3	The Main Result	72
6.4	Einstein’s Spacetime Geometry	74
6.5	Three-Dimensional Hyperbolic Geometry	79
7	Exercises for Act II	83
ACT III		
Curvature		
8	Curvature of Plane Curves	97
8.1	Introduction	97
8.2	The Circle of Curvature	98
8.3	Newton’s Curvature Formula	100
8.4	Curvature as Rate of Turning	101
8.5	Example: Newton’s <i>Tractrix</i>	104
9	Curves in 3-Space	106
10	The Principal Curvatures of a Surface	109
10.1	Euler’s Curvature Formula	109
10.2	Proof of Euler’s Curvature Formula	110
10.3	Surfaces of Revolution	112
11	Geodesics and Geodesic Curvature	115
11.1	Geodesic Curvature and Normal Curvature	115
11.2	Meusnier’s Theorem	117
11.3	Geodesics are “Straight”	118
11.4	Intrinsic Measurement of Geodesic Curvature	119
11.5	A Simple Extrinsic Way to Measure Geodesic Curvature	120
11.6	A New Explanation of the Sticky-Tape Construction of Geodesics	120
11.7	Geodesics on Surfaces of Revolution	121
11.7.1	Clairaut’s Theorem on the Sphere	121
11.7.2	Kepler’s Second Law	123
11.7.3	Newton’s Geometrical Demonstration of Kepler’s Second Law	124
11.7.4	Dynamical Proof of Clairaut’s Theorem	126
11.7.5	Application: Geodesics in the Hyperbolic Plane (Revisited)	128

12	The Extrinsic Curvature of a Surface	130
12.1	Introduction	130
12.2	The Spherical Map	130
12.3	Extrinsic Curvature of Surfaces	131
12.4	What Shapes Are Possible?	135
13	Gauss's <i>Theorema Egregium</i>	138
13.1	Introduction	138
13.2	Gauss's <i>Beautiful Theorem</i> (1816)	138
13.3	Gauss's <i>Theorema Egregium</i> (1827)	140
14	The Curvature of a Spike	143
14.1	Introduction	143
14.2	Curvature of a Conical Spike	143
14.3	The Intrinsic and Extrinsic Curvature of a Polyhedral Spike	145
14.4	<i>The Polyhedral Theorema Egregium</i>	147
15	The Shape Operator	149
15.1	Directional Derivatives	149
15.2	The Shape Operator S	151
15.3	The Geometric Effect of S	152
15.4	DETOUR: The <i>Geometry</i> of the Singular Value Decomposition and of the Transpose	154
15.5	The General Matrix of S	158
15.6	Geometric Interpretation of S and Simplification of $[S]$	159
15.7	$[S]$ Is Completely Determined by Three Curvatures	161
15.8	Asymptotic Directions	162
15.9	Classical Terminology and Notation: <i>The Three Fundamental Forms</i>	164
16	Introduction to the Global Gauss–Bonnet Theorem	165
16.1	Some Topology and the Statement of the Result	165
16.2	Total Curvature of the Sphere and of the Torus	168
16.2.1	Total Curvature of the Sphere	168
16.2.2	Total Curvature of the Torus	169
16.3	Seeing $\mathcal{K}(\mathcal{S}_g)$ via a Thick Pancake	170
16.4	Seeing $\mathcal{K}(\mathcal{S}_g)$ via Bagels and Bridges	171
16.5	The Topological Degree of the Spherical Map	172
16.6	Historical Note	174
17	First (Heuristic) Proof of the Global Gauss–Bonnet Theorem	175
17.1	Total Curvature of a Plane Loop: Hopf's <i>Umlaufsatz</i>	175
17.2	Total Curvature of a Deformed Circle	178
17.3	Heuristic Proof of Hopf's <i>Umlaufsatz</i>	179

17.4	Total Curvature of a Deformed Sphere	180
17.5	Heuristic Proof of the Global Gauss–Bonnet Theorem	181
18	Second (Angular Excess) Proof of the Global Gauss–Bonnet Theorem	183
18.1	The Euler Characteristic	183
18.2	Euler’s (Empirical) Polyhedral Formula	183
18.3	Cauchy’s Proof of Euler’s Polyhedral Formula	186
18.3.1	Flattening Polyhedra	186
18.3.2	The Euler Characteristic of a Polygonal Net	187
18.4	Legendre’s Proof of Euler’s Polyhedral Formula	188
18.5	Adding Handles to a Surface to Increase Its Genus	190
18.6	Angular Excess Proof of the Global Gauss–Bonnet Theorem	193
19	Third (Vector Field) Proof of the Global Gauss–Bonnet Theorem	195
19.1	Introduction	195
19.2	Vector Fields in the Plane	195
19.3	The Index of a Singular Point	196
19.4	The Archetypal Singular Points: Complex Powers	198
19.5	Vector Fields on Surfaces	201
19.5.1	The Honey-Flow Vector Field	201
19.5.2	Relation of the Honey-Flow to the Topographic Map	203
19.5.3	Defining the Index on a Surface	204
19.6	The Poincaré–Hopf Theorem	206
19.6.1	Example: The Topological Sphere	206
19.6.2	Proof of the Poincaré–Hopf Theorem	207
19.6.3	Application: Proof of the Euler–L’Huillier Formula	208
19.6.4	Poincaré’s Differential Equations Versus Hopf’s <i>Line Fields</i>	210
19.7	Vector Field Proof of the Global Gauss–Bonnet Theorem	214
19.8	The Road Ahead	218
20	Exercises for Act III	219

ACT IV

Parallel Transport

21	An Historical Puzzle	231
22	Extrinsic Constructions	233
22.1	Project into the Surface as You Go!	233
22.2	Geodesics and Parallel Transport	235
22.3	Potato-Peeler Transport	236
23	Intrinsic Constructions	240
23.1	Parallel Transport via Geodesics	240
23.2	The Intrinsic (aka, “Covariant”) Derivative	241

24	Holonomy	245
24.1	Example: The Sphere	245
24.2	Holonomy of a General Geodesic Triangle	246
24.3	Holonomy Is Additive	248
24.4	Example: The Hyperbolic Plane	248
25	An Intuitive Geometric Proof of the <i>Theorema Egregium</i>	252
25.1	Introduction	252
25.2	Some Notation and Reminders of Definitions	253
25.3	The Story So Far	253
25.4	The Spherical Map Preserves Parallel Transport	254
25.5	The Beautiful Theorem and <i>Theorema Egregium</i> Explained	256
26	Fourth (Holonomy) Proof of the Global Gauss–Bonnet Theorem	257
26.1	Introduction	257
26.2	Holonomy Along an <i>Open</i> Curve?	257
26.3	Hopf's Intrinsic Proof of the Global Gauss–Bonnet Theorem	258
27	Geometric Proof of the Metric Curvature Formula	261
27.1	Introduction	261
27.2	The Circulation of a Vector Field Around a Loop	262
27.3	Dry Run: Holonomy in the Flat Plane	264
27.4	Holonomy as the Circulation of a Metric-Induced Vector Field in the Map	266
27.5	Geometric Proof of the Metric Curvature Formula	268
28	Curvature as a Force between Neighbouring Geodesics	269
28.1	Introduction to the Jacobi Equation	269
28.1.1	Zero Curvature: The Plane	269
28.1.2	Positive Curvature: The Sphere	270
28.1.3	Negative Curvature: The Pseudosphere	272
28.2	Two Proofs of the Jacobi Equation	274
28.2.1	Geodesic Polar Coordinates	274
28.2.2	Relative Acceleration = Holonomy of Velocity	276
28.3	The Circumference and Area of a Small Geodesic Circle	278
29	Riemann's Curvature	280
29.1	Introduction and Summary	280
29.2	Angular Excess in an n -Manifold	281
29.3	Parallel Transport: Three Constructions	282
29.3.1	Closest Vector on Constant-Angle Cone	282
29.3.2	Constant Angle within a Parallel-Transported Plane	283
29.3.3	<i>Schild's Ladder</i>	284

29.4	The Intrinsic (aka "Covariant") Derivative $\nabla_{\mathbf{v}}$	284
29.5	The Riemann Curvature Tensor	286
29.5.1	Parallel Transport Around a Small "Parallelogram"	286
29.5.2	Closing the "Parallelogram" with the Vector Commutator	287
29.5.3	The General Riemann Curvature Formula	288
29.5.4	Riemann's Curvature Is a <i>Tensor</i>	291
29.5.5	Components of the Riemann Tensor	292
29.5.6	For a Given w_0 , the Vector Holonomy <i>Only</i> Depends on the <i>Plane</i> of the Loop and Its <i>Area</i>	293
29.5.7	Symmetries of the Riemann Tensor	294
29.5.8	Sectional Curvatures	296
29.5.9	Historical Notes on the Origin of the Riemann Tensor	297
29.6	The Jacobi Equation in an n -Manifold	299
29.6.1	Geometrical Proof of the Sectional Jacobi Equation	299
29.6.2	Geometrical Implications of the Sectional Jacobi Equation	300
29.6.3	Computational Proofs of the Jacobi Equation and the Sectional Jacobi Equation	301
29.7	The Ricci Tensor	302
29.7.1	Acceleration of the Area Enclosed by a Bundle of Geodesics	302
29.7.2	Definition and Geometrical Meaning of the Ricci Tensor	304
29.8	Coda	306
30	Einstein's Curved Spacetime	307
30.1	Introduction: " <i>The Happiest Thought of My Life.</i> "	307
30.2	Gravitational Tidal Forces	308
30.3	Newton's Gravitational Law in Geometrical Form	312
30.4	The Spacetime Metric	314
30.5	Spacetime Diagrams	315
30.6	Einstein's Vacuum Field Equation in Geometrical Form	317
30.7	The Schwarzschild Solution and the First Tests of the Theory	319
30.8	Gravitational Waves	323
30.9	The Einstein Field Equation (with Matter) in Geometrical Form	326
30.10	Gravitational Collapse to a Black Hole	329
30.11	The Cosmological Constant: " <i>The Greatest Blunder of My Life.</i> "	331
30.12	The End	333
31	Exercises for Act IV	334

ACT V
Forms

32	1-Forms	345
32.1	Introduction	345
32.2	Definition of a 1-Form	346
32.3	Examples of 1-Forms	347
32.3.1	Gravitational Work	347
32.3.2	Visualizing the Gravitational Work 1-Form	348
32.3.3	Topographic Maps and the Gradient 1-Form	349
32.3.4	Row Vectors	352
32.3.5	Dirac's Bras	352
32.4	Basis 1-Forms	352
32.5	Components of a 1-Form	354
32.6	The Gradient as a 1-Form: df	354
32.6.1	Review of the Gradient as a Vector: ∇f	354
32.6.2	The Gradient as a 1-Form: df	355
32.6.3	The Cartesian 1-Form Basis: $\{dx^j\}$	356
32.6.4	The 1-Form Interpretation of $df = (\partial_x f) dx + (\partial_y f) dy$	357
32.7	Adding 1-Forms Geometrically	357
33	Tensors	360
33.1	Definition of a Tensor: Valence	360
33.2	Example: Linear Algebra	361
33.3	New Tensors from Old	361
33.3.1	Addition	361
33.3.2	Multiplication: The Tensor Product	361
33.4	Components	362
33.5	Relation of the Metric Tensor to the Classical Line Element	363
33.6	Example: Linear Algebra (Again)	364
33.7	Contraction	365
33.8	Changing Valence with the Metric Tensor	366
33.9	Symmetry and Antisymmetry	368
34	2-Forms	370
34.1	Definition of a 2-Form and of a p -Form	370
34.2	Example: The Area 2-Form	371
34.3	The Wedge Product of Two 1-Forms	372
34.4	The Area 2-Form in Polar Coordinates	374
34.5	Basis 2-Forms and Projections	375
34.6	Associating 2-Forms with Vectors in \mathbb{R}^3 : Flux	376
34.7	Relation of the Vector and Wedge Products in \mathbb{R}^3	379
34.8	The Faraday and Maxwell Electromagnetic 2-Forms	381

35	3-Forms	386
35.1	A 3-Form Requires Three Dimensions	386
35.2	The Wedge Product of a 2-Form and 1-Form	386
35.3	The Volume 3-Form	387
35.4	The Volume 3-Form in Spherical Polar Coordinates	388
35.5	The Wedge Product of Three 1-Forms and of p 1-Forms	389
35.6	Basis 3-Forms	390
35.7	Is $\Psi \wedge \Psi \neq 0$ Possible?	391
36	Differentiation	392
36.1	The Exterior Derivative of a 1-Form	392
36.2	The Exterior Derivative of a 2-Form and of a p -Form	394
36.3	The Leibniz Rule for Forms	394
36.4	Closed and Exact Forms	395
36.4.1	A Fundamental Result: $d^2 = 0$	395
36.4.2	Closed and Exact Forms	396
36.4.3	Complex Analysis: Cauchy–Riemann Equations	397
36.5	Vector Calculus via Forms	398
36.6	Maxwell’s Equations	401
37	Integration	404
37.1	The Line Integral of a 1-Form	404
37.1.1	Circulation and Work	404
37.1.2	Path-Independence \iff Vanishing Loop Integrals	405
37.1.3	The Integral of an Exact Form: $\varphi = df$	406
37.2	The Exterior Derivative as an Integral	406
37.2.1	$d(1\text{-Form})$	406
37.2.2	$d(2\text{-Form})$	409
37.3	Fundamental Theorem of Exterior Calculus (Generalized Stokes’s Theorem)	411
37.3.1	Fundamental Theorem of Exterior Calculus	411
37.3.2	Historical Aside	411
37.3.3	Example: Area	412
37.4	The Boundary of a Boundary Is Zero!	412
37.5	The Classical Integral Theorems of Vector Calculus	413
37.5.1	$\Phi = 0\text{-Form}$	413
37.5.2	$\Phi = 1\text{-Form}$	414
37.5.3	$\Phi = 2\text{-Form}$	415
37.6	Proof of the Fundamental Theorem of Exterior Calculus	415
37.7	Cauchy’s Theorem	417
37.8	The Poincaré Lemma for 1-Forms	418
37.9	A Primer on de Rham Cohomology	419
37.9.1	Introduction	419

37.9.2	A Special 2-Dimensional Vortex Vector Field	419
37.9.3	The Vortex 1-Form Is Closed	420
37.9.4	Geometrical Meaning of the Vortex 1-Form	420
37.9.5	The Topological Stability of the Circulation of a Closed 1-Form	421
37.9.6	The First de Rham Cohomology Group	423
37.9.7	The Inverse-Square Point Source in \mathbb{R}^3	424
37.9.8	The Second de Rham Cohomology Group	426
37.9.9	The First de Rham Cohomology Group of the Torus	428

38 Differential Geometry via Forms 430

38.1	Introduction: Cartan's Method of Moving Frames	430
38.2	Connection 1-Forms	432
38.2.1	Notational Conventions and Two Definitions	432
38.2.2	Connection 1-Forms	432
38.2.3	WARNING: Notational Hazing Rituals Ahead!	434
38.3	The Attitude Matrix	435
38.3.1	The Connection Forms via the Attitude Matrix	435
38.3.2	Example: The Cylindrical Frame Field	436
38.4	Cartan's Two Structural Equations	438
38.4.1	The Duals θ^i of \mathbf{m}_i in Terms of the Duals dx^j of \mathbf{e}_j	438
38.4.2	Cartan's First Structural Equation	439
38.4.3	Cartan's Second Structural Equation	440
38.4.4	Example: The Spherical Frame Field	441
38.5	The Six Fundamental Form Equations of a Surface	446
38.5.1	Adapting Cartan's Moving Frame to a Surface: The Shape Operator and the Extrinsic Curvature	446
38.5.2	Example: The Sphere	447
38.5.3	Uniqueness of Basis Decompositions	447
38.5.4	The Six Fundamental Form Equations of a Surface	448
38.6	Geometrical Meanings of the Symmetry Equation and the Peterson–Mainardi–Codazzi Equations	449
38.7	Geometrical Form of the Gauss Equation	450
38.8	Proof of the Metric Curvature Formula and the <i>Theorema Egregium</i>	451
38.8.1	Lemma: Uniqueness of ω_{12}	451
38.8.2	Proof of the Metric Curvature Formula	451
38.9	A New Curvature Formula	452
38.10	Hilbert's Lemma	453
38.11	Liebmann's Rigid Sphere Theorem	454
38.12	The Curvature 2-Forms of an n-Manifold	455
38.12.1	Introduction and Summary	455
38.12.2	The Generalized Exterior Derivative	457

38.12.3	Extracting the Riemann Tensor from the Curvature 2-Forms	459
38.12.4	The Bianchi Identities Revisited	459
38.13	The Curvature of the Schwarzschild Black Hole	460
39	Exercises for Act V	465
	<i>Further Reading</i>	475
	<i>Bibliography</i>	485
	<i>Index</i>	491