

A Relativist's Toolkit

Paperback Re-issue

This textbook fills a gap in the literature on general relativity by providing the advanced student with practical tools for the computation of many physically interesting quantities. The context is provided by the mathematical theory of black holes: one of the most elegant, successful, and relevant applications of general relativity.

Among the topics discussed are congruences of timelike and null geodesics, the embedding of spacelike, timelike, and null hypersurfaces in spacetime, and the Lagrangian and Hamiltonian formulations of general relativity. The book also covers the application of null congruences to the description of the event horizon, how integration on a null hypersurface relates to black-hole mechanics, and the relationship between the gravitational Hamiltonian and a black hole's mass and angular momentum.

Although the book is self-contained, it is not meant to serve as an introduction to general relativity. Instead, it is meant to help the reader acquire advanced skills and become a competent researcher in relativity and gravitational physics. The primary readership consists of graduate students in gravitational physics. The book will also be a useful reference for more seasoned researchers working in this field.

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