

This book is intended to provide a working knowledge of topics in exterior differential forms, differential geometry, algebraic and differential topology, Lie groups, fiber and vector bundles, and Chern forms, that are helpful for a deeper understanding of classical and modern physics and engineering.

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## A Few Questions Considered in The Geometry of Physics

1. How is the **air pressure** in an irregular soap bubble related to its curvature? Which curvature? (p. 227)
2. How does the observed fact that there are nearby thermodynamic states that cannot be connected adiabatically imply the existence of **entropy**, and why does entropy increase? (p. 183)
3. How does special relativity show that the **magnetic flux law**  $\text{div } \mathbf{B}=0$  **implies Faraday's law**, and that **Gauss' law implies Ampere-Maxwell's**? (p.200)
4. How does **Weyl's "principle of gauge invariance"** lead to the **conservation of electric charge in quantum theory**? (p. 536)
5. How does algebraic topology influence whether one can **maintain an electric current** in a closed wire loop that sits in a curved three-dimensional space? (p.122), and how does the **topology of a configuration space** influence the existence of periodic motions in a dynamical system? (pp. 284 and 331)
6. Gauss invented "intrinsic" curvature (p. 232) and equated it to his "extrinsic" curvature for a surface in Euclidean space; how does Einstein's **general relativity** generalize this? (p. 318)
7. How are properties of **fluid flows** (Euler's equations, circulation, vorticity, Woltjer's theorem of magnetohydrodynamics) described via the Lie derivative? (p.144)
8. In what sense is a full rotation about an axis "something," whereas two full rotations is "nothing," and how is this related to **Dirac's equation**? (pp.499 and 517)
9. What was the original **quark model** of elementary particle physics and how does Lie group theory relate the **masses of the pion, eta, and kaon mesons** in this model? (p.651)

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