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Topological, Differential and Conformal Geometry of Surfaces

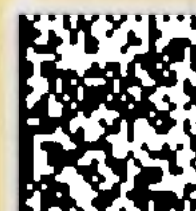
This book provides an introduction to the main geometric structures that are carried by compact surfaces, with an emphasis on the classical theory of Riemann surfaces. It first covers the prerequisites, including the basics of differential forms, the Poincaré Lemma, the Morse Lemma, the classification of compact connected oriented surfaces, Stokes' Theorem, fixed point theorems and rigidity theorems. There is also a novel presentation of planar hyperbolic geometry. Moving on to more advanced concepts, it covers topics such as Riemannian metrics, the isometric torsion-free connection on vector fields, the Ansatz of Koszul, the Gauss–Bonnet Theorem, and integrability. These concepts are then used for the study of Riemann surfaces. One of the focal points is the Uniformization Theorem for compact surfaces, an elementary proof of which is given via a property of the energy functional. Among numerous other results, there is also a proof of Chow's Theorem on compact holomorphic submanifolds in complex projective spaces.

Based on lecture courses given by the author, the book will be accessible to undergraduates and graduates interested in the analytic theory of Riemann surfaces.

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