

Map of the World

An Introduction to Mathematical Geodesy

Carl Friedrich Gauss, the "foremost of mathematicians," was a land surveyor. Measuring and calculating geodetic networks on the curved Earth was the inspiration for some of his greatest mathematical discoveries. This is just one example of how mathematics and geodesy, the science and art of measuring and mapping our world, have evolved together throughout history.

This text is for students and professionals in geodesy, land surveying, and geospatial science who need to understand the mathematics of describing the Earth and capturing her in maps and geospatial data: the discipline known as mathematical geodesy. ***Map of the World: An Introduction to Mathematical Geodesy*** aims to provide an accessible introduction to this area, presenting and developing the mathematics relating to maps, mapping, and the production of geospatial data. Described are the theory and its fundamental concepts, its application for processing, analyzing, transforming, and projecting geospatial data, and how these are used in producing charts and atlases. Also touched upon are the multitude of cross-overs into other sciences sharing in the adventure of discovering what our world really looks like.

FEATURES

- Written in a fluid and accessible style, replete with exercises; adaptable for courses on different levels.
- Suitable for students and professionals in the mapping sciences, but also for lovers of maps and map making.

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Preface	xi
Acknowledgments	xiii
Acronyms	xv
1 A brief history of mapping	1
1.1 Map and scale	2
1.2 Gerardus Mercator and Mercator's map of the world	3
1.3 Construction of the Mercator projection	5
1.4 Loxodromes or rhumb lines	7
1.5 A political map of the world?	8
1.6 Towards the precise figure of the Earth	9
1.6.1 Rotation and flattening of the Earth	10
1.6.2 The big project of the French Academy of Sciences	13
Exercises	14
2 Popular conformal map projections	17
2.1 The Lambert projection	19
2.2 About the isometric latitude	22
2.3 The Mercator projection	23
2.4 The stereographic projection	24
Exercises	28
3 The complex plane and conformal mappings	29
3.1 Complex numbers	29
3.2 The polar representation of complex numbers	31
3.3 On the nature of the exponential function	33
3.4 Euler's formula	35
3.5 De Moivre's formula	36
3.6 The logarithm function	37
Exercises	39

4	Complex analysis	43
4.1	Limit and continuity of a complex function	43
4.2	The complex derivative and analytic functions	45
4.3	The Cauchy–Riemann conditions	46
4.4	Series expansions	48
4.5	An analytic function as a power series	49
4.6	Complex Taylor series	51
	Exercises	52
5	Conformal mappings	53
5.1	Helmert mappings or transformations	54
5.2	Möbius transformations	57
5.3	Möbius transformations and the stereographic projection . .	62
5.4	Numerical conformal mappings	64
	Exercises	65
6	Transversal Mercator projections	67
6.1	Map projections used in Finland	67
6.1.1	The historical Gauss–Krüger and KKJ	67
6.1.2	Modern map projections Gauss–Krüger and UTM . . .	70
6.2	Gauss–Krüger projection computations	72
6.2.1	Analyticity and the Cauchy–Riemann conditions	73
6.2.2	Progress of the computation	74
7	Spherical trigonometry	77
7.1	Spherical excess	77
7.2	Surface area of a triangle on a sphere	79
7.3	Rectangular spherical triangle	80
7.4	General spherical triangle	80
7.5	Deriving the equations using vectors in space	82
7.6	Polarization	84
7.7	Solving the spherical triangle with the method of additaments	86
7.8	Solving the spherical triangle using Legendre’s method	87
7.9	The half-angle cosine rule	87
7.10	The geodetic forward problem on the sphere	88
7.11	The geodetic inverse problem on the sphere	89
	Exercises	89

8	The geometry of the ellipsoid of revolution	91
8.1	The geodesic as a solution of a system of differential equations	92
8.2	An interesting invariant	93
8.3	The geodetic forward problem	94
8.4	The geodetic inverse problem	95
8.5	Representations of sphere and ellipsoid	96
8.6	Different types of latitude and relations between them	97
8.7	Different measures for the flattening	99
8.8	Co-ordinates on the meridian ellipse	99
8.9	Three-dimensional rectangular co-ordinates on the reference ellipsoid	100
8.10	Computing geodetic co-ordinates from rectangular ones . . .	101
8.11	Length of the meridian arc	103
8.12	The reference ellipsoid and the gravity field	103
	Exercises	107
9	Three-dimensional co-ordinates and transformations	109
9.1	About rotations and rotation matrices	110
9.2	Chaining matrices in three dimensions	111
9.3	Orthogonal matrices	112
9.4	Topocentric systems	114
9.4.1	Geocentric to topocentric and back	115
9.5	Solving the parameters of a three-dimensional Helmert transfor- mation	118
9.6	Variants of the Helmert transformation	122
9.7	The geodetic forward or inverse problem with rotation matrices	123
9.7.1	The geodetic forward problem	124
9.7.2	The geodetic inverse problem	125
9.7.3	Comparison with the ellipsoidal surface solution	125
	Exercises	126
10	Co-ordinate reference systems	127
10.1	History	127
10.2	Parameters of the GRS80 system	128
10.3	GRS80 and modern co-ordinate reference systems	130
10.4	Co-ordinate reference systems and their realizations	132
10.4.1	Terminology	132
10.4.2	The practical viewpoint	132
10.4.3	The ITRS/ETRS systems and their realizations	134

10.5	Transformations between different ITRFs	136
10.6	The Finnish ETRS89 realization, EUREF-FIN	137
10.7	A triangle-wise affine transformation for the Finnish territory	138
	Exercises	142
11	Co-ordinates of heaven and Earth	143
11.1	Orientation of the Earth	145
11.2	Polar motion	146
11.3	Geocentric systems	147
11.3.1	Geocentric terrestrial systems	148
11.3.2	The conventional terrestrial system	149
11.3.3	Instantaneous terrestrial system	150
11.3.4	The geocentric quasi-inertial system	150
11.4	Sidereal time	152
11.5	Celestial trigonometry	154
11.6	Rotation matrices and celestial co-ordinate systems	157
	Exercises	158
12	The orbital motion of satellites	161
12.1	Kepler's laws	161
12.2	The circular orbit approximation in manual computation	168
12.3	Crossing a certain latitude in the inertial system	169
12.4	Topocentric co-ordinates of the satellite	171
12.5	Latitude crossing in the terrestrial system	173
12.6	Determining the orbit from observations	173
	Exercises	179
13	The surface theory of Gauss	181
13.1	A curve in space	182
13.2	The first fundamental form (metric)	184
13.3	The second fundamental form	186
13.4	Principal curvatures of a surface	188
13.5	A curve on a surface	192
13.5.1	Tangent vector	192
13.5.2	Curvature vector	192
13.6	The geodesic	195
13.6.1	Describing the geodesic interiorly, in surface co-ordinates	196
13.6.2	Describing the geodesic exteriorly, using vectors in space	196
13.7	More generally on surface theory	198
	Exercises	199

14.1	What is a tensor?	205
14.1.1	Vectors	205
14.1.2	Tensors	206
14.1.3	The geometric presentation of a tensor and its invariants	207
14.1.4	Tensors in general co-ordinates	208
14.1.5	Trivial tensors	209
14.2	The metric tensor	210
14.3	The inverse metric tensor	213
14.3.1	Raising or lowering tensor indices	213
14.3.2	The eigenvalues and eigenvectors of a tensor	214
14.3.3	Visualization of a tensor	214
14.4	Parallellity of vectors and parallel transport	214
14.5	The Christoffel symbols	215
14.6	Alternative equations for the geodesic	218
14.7	The curvature tensor	219
	Exercises	222

15 Map projections in light of surface theory

227

15.1	The Gauss total curvature and spherical excess	227
15.2	Curvature in quasi-Cartesian co-ordinates	232
15.3	Map projections and scale	234
15.3.1	On the Earth's surface	234
15.3.2	In the map plane	235
15.3.3	The scale	235
15.3.4	The Tissot indicatrix	238
15.4	Curvature of the Earth's surface and scale	239
	Exercises	241

A The stereographic projection maps circles into circles

245

B Isometric latitude on the ellipsoid

247

C Useful relations between the principal radii of curvature

249

D Christoffel's symbols from the metric

251

E	The Riemann tensor from the Christoffel symbols	253
F	Conformal mappings in spaces of any dimension	255
	References	257
	Index	265