

DISCRETE MATHEMATICS AND ITS APPLICATIONS

A *Choice* journal "Outstanding Academic Title," the first edition of this bestseller was lauded for its detailed yet engaging treatment of permutations. Providing more than enough material for a one-semester course, **Combinatorics of Permutations, Third Edition**, continues to clearly show the usefulness of this subject for both students and researchers.

The research in combinatorics of permutations has advanced rapidly since this book was published in a first edition. Now the third edition offers not only updated results, but it also remains the leading textbook for a course on the topic.

Coverage is mostly enumerative, but there are algebraic, analytic, and topological parts as well, and applications.

Since the publication of the second edition, there is tremendous progress in pattern avoidance (Chapters 4 and 5). There is also significant progress in the analytic combinatorics of permutations, which will be incorporated.

- A completely new technique from extremal combinatorics disproved a long-standing conjecture, and this is presented in Chapter 4.
- The area of universal permutations has undergone a lot of very recent progress, and that has been noticed outside the academic community as well. This also influenced the revision of Chapter 5.
- New results in stack sorting are added to Chapter 8.
- Chapter 9 applications to biology have been revised.

The author's other works include *Introduction to Enumerative and Analytic Combinatorics, Second Edition* (*Choice* "Outstanding Academic Title") and *Handbook of Enumerative Combinatorics*, published by CRC Press. The author also serves as Series Editor of CRC's *Discrete Mathematics and Its Applications*.

MATHEMATICS



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