This monograph is an intertwined tale of eigenvalues and their use in unlocking a thousand secrets about graphs. The stories will be told—how the spectrum reveals fundamental properties of a graph, how spectral graph theory links the discrete universe to the continuous one through geometric, analytic and algebraic techniques, and how, through eigenvalues, theory and applications in communications and computer science come together in symbiotic harmony.

—from the Preface

Beautifully written and elegantly presented, this book is based on 10 lectures given at the CBMS workshop on spectral graph theory in June 1994 at Fresno State University. Chung's well-written exposition can be likened to a conversation with a good teacher—one who not only gives you the facts, but tells you what is really going on, why it is worth doing, and how it is related to familiar ideas in other areas. The monograph is accessible to the nonexpert who is interested in reading about this evolving area of mathematics.

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